BASE SIZES FOR FINITE UNITARY AND SYMPLECTIC GROUPS WITH SOLVABLE STABILISERS

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Abstract. Let G be a transitive permutation group on a finite set with solvable point stabiliser. In 2010, Vdovin conjectured that the base size of G is at most 5. Burness proved this conjecture for primitive G. The problem was reduced by Vdovin in 2012 to the case when G is an almost simple group, and reduced to groups of Lie type by Baykalov and Burness. This is the second paper of the series devoted to the study of Vodvin's conjecture for classical groups. In the first paper, we prove a strong form of the conjecture for almost simple groups with socle isomorphic to $\mathrm{PSL}_n(q)$. In the present paper we extend this result to almost simple groups with socle isomorphic to $\mathrm{PSU}_n(q)$ and $\mathrm{PSp}_n(q)$. The final paper will establish the conjecture for orthogonal groups. Together, these three paper will complete the proof of Vdovin's conjecture for all almost simple classical groups.

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1. Introduction

1.1. **Main problem and results.** Consider a permutation group G on a finite set Ω. A **base** of G is a subset of Ω such that its pointwise stabiliser is trivial. The **base size** of G is the minimal size of a base. The study of bases and base sizes of permutation groups is an active research area with a rich history. These concepts have applications in abstract group theory and computational algebra; for a summary, see [2] and [13]. In a case of a transitive group G with point stabiliser H, we write $b_H(G)$ for the base size, noting that the action of G on Ω is permutation isomorphic to the action of G on right H-cosets by right multiplication.

This paper is the second in a series of three devoted to the study of base sizes of finite transitive permutation groups with solvable point stabilisers. In [29, Problem 17.41 b)], Vdovin conjectures (in a slightly different notation) that $b_S(G) \leq 5$ for a transitive permutation group G with a solvable point stabiliser S. In [14] Burness proved this conjecture in the case of primitive G. We mention that Seress [35] established $b_S(G) \leq 4$ in the case of solvable primitive G, so the maximal subgroup S, of course, is solvable as well. Notice that the bound in the conjecture is the best possible since $b_S(G) = 5$ if $G = \mathrm{Sym}(8)$ and $S = \mathrm{Sym}(4) \wr \mathrm{Sym}(2)$. This can be easily verified. In fact, there are infinitely many examples with $b_S(G) = 5$, for example see [14, Remark 8.3].

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Vdovin reduces the above conjecture to the case of almost simple group G in [38, Theorem 1]. In order to explain how exactly the reduction works, let us define $\mathrm{Reg}(G, k)$ to be the number of distinct regular orbits in the action of a permutation group $G \le \mathrm{Sym}(\Omega)$ on Ω^k; here G acts on Ω^k by

$$(\alpha_1, \dots, \alpha_k)g = (\alpha_1 g, \dots, \alpha_k g).$$

Analogous to $b_H(G)$, we write $\mathrm{Reg}_H(G, k)$ for $\mathrm{Reg}(G, k)$ in the case when Ω is the set of right cosets of a subgroup H. Since a regular point in Ω^k forms a base for G acting on Ω, $\mathrm{Reg}_H(G, k) = 0$ for $k < b_H(G)$. Now we return to the reduction: [38, Theorem 1] implies that, in order to prove Vdovin's conjecture, it is sufficient to show

$$\mathrm{Reg}_S(G, 5) \ge 5$$

for every almost simple group G and each of its maximal solvable subgroups S; that is, S is not contained in any larger solvable subgroup of G. This inequality is established for almost simple groups with alternating and sporadic socle in [3] and [15] respectively. Therefore, Vdovin's conjecture is now reduced to the case of almost simple group of Lie type. Our paper [5] is the first to consider the general form of the conjecture for groups of Lie type. There we prove the above inequality for all maximal solvable subgroups S of an almost simple group G with socle isomorphic to $\mathrm{PSL}_n(q)$.

In this paper, we consider almost simple groups groups with socle isomorphic to $\mathrm{PSU}_n(q)$ or $\mathrm{PSp}_n(q)'$ and prove the following.

Theorem 1.1. *Let G_0 be a classical finite simple group of Lie type with socle isomorphic to either $\mathrm{PSU}_n(q)$ or $\mathrm{PSp}_n(q)'$. If S is a maximal solvable subgroup of $\mathrm{Aut}(G_0)$, then $\mathrm{Reg}_S(S \cdot G_0, 5) \ge 5$. In particular, $b_S(S \cdot G_0) \le 5$.*

We briefly summarise the structure of an almost simple group with socle $\mathrm{PSU}_n(q)$ or $\mathrm{PSp}_n(q)'$ (see [21] for details). Then we state more detailed results which also illustrate the cases that arise in the proof.

Since $\mathrm{PSU}_2(q) \cong \mathrm{PSp}_2(q) \cong \mathrm{PSL}_2(q)$, we only need to consider $\mathrm{PSU}_n(q)$ for $n \ge 3$ and $\mathrm{PSp}_n(q)'$ for $n \ge 4$. Note that $\mathrm{PSp}_n(q) = \mathrm{PSp}_n(q)'$ is simple for $n \ge 4$ unless $(n, q) = (4, 2)$ where the simple subgroup $\mathrm{PSp}_4(2)'$ has index 2 in $\mathrm{PSp}_4(2)$. Let $\Gamma\mathrm{U}_n(q)$ and $\Gamma\mathrm{Sp}_n(q)$ be the groups of semisimilarities of a non-degenerate n-dimensional unitary and symplectic spaces over $\mathbb{F}_{q^2}$ and $\mathbb{F}_q$ respectively (see Section 2.1 for details) and let Z be the group of all scalar matrices in the corresponding group. If $\mathrm{PSU}_n(q)$ is simple, so $(n, q) \ne (3, 2)$, then

$$\mathrm{Aut}(\mathrm{PSU}_n(q)) \cong \mathrm{P}\Gamma\mathrm{U}_n(q) = \Gamma\mathrm{U}_n(q)/Z.$$

Unless $(n, q) = (4, 2^f)$ for some $f \ge 1$,

$$\mathrm{Aut}(\mathrm{PSp}_n(q)) \cong \mathrm{P}\Gamma\mathrm{Sp}_n(q) = \Gamma\mathrm{Sp}_n(q)/Z.$$

If $(n, q) = (4, 2^f)$, then $\mathrm{PSp}_4(q)'$ has a graph-field automorphism of order $2f$; see [19, §12.3] for details.

We establish Theorem 1.1 by proving the following three statements.

Theorem B1. *Let $n \ge 3$ and (n, q) is not $(3, 2)$. If S is a maximal solvable subgroup of $\mathrm{P}\Gamma\mathrm{U}_n(q)$, then one of the following holds:*

1. *$b_S(S \cdot \mathrm{PSU}_n(q)) \le 4$, so $\mathrm{Reg}_S(S \cdot \mathrm{PSU}_n(q), 5) \ge 5$;*
2. *$(n, q) = (5, 2)$ and S is the stabiliser in $\mathrm{P}\Gamma\mathrm{U}_n(q)$ of a totally isotropic 1-dimensional subspace of the natural module, $b_S(S \cdot \mathrm{PSU}_n(q)) = 5$ and $\mathrm{Reg}_S(S \cdot \mathrm{PSU}_n(q), 5) \ge 5$.*

Theorem C1. *Let $n \geq 4$. If S is a maximal solvable subgroup of* $\mathrm{P\Gamma Sp}_n(q)$*, then* $b_S(S \cdot \mathrm{PSp}_n(q)) \leq 4$*, so* $\mathrm{Reg}_S(S \cdot \mathrm{PSp}_n(q), 5) \geq 5$.

Theorem C2. *Let q be even and let* $A = \mathrm{Aut}(\mathrm{PSp}_4(q)')$*. If $S \leq A$ is a maximal solvable subgroup, then* $b_S(S \cdot \mathrm{PSp}_4(q)') \leq 4$*, so* $\mathrm{Reg}_S(S \cdot \mathrm{PSp}_n(q)', 5) \geq 5$.

Theorem 1.1 now follows from Theorems B1, C1 and C2 via the following lemma proved in [5, Section 2.1].

Lemma 1.2. *Let G_0 be a finite simple nonabelian group. Let* $G_0 \leq G \leq \mathrm{Aut}(G_0)$ *and let $S \leq G$ be solvable. If* $\mathrm{Reg}_H(H \cdot G_0, 5) \geq 5$ *for every maximal solvable subgroups H of* $\mathrm{Aut}(G_0)$*, then* $\mathrm{Reg}_S(G, 5) \geq 5$*, for every solvable subgroup S of G.*

1.2. **Methods and ideas.** It is convenient for us to work with groups of matrices which makes the action on the set of right cosets of a subgroup not faithful in most cases. It is possible to extend the notion of $b_H(G)$ and $\mathrm{Reg}_H(G, k)$ to an arbitrary finite group G by

$$b_H(G) := b(G/H_G) \text{ and } \mathrm{Reg}_H(G, k) := \mathrm{Reg}(G/H_G, k).$$

Here $H_G = \cap_{g\in G} H^g$ and the action of G/H_G on the set Ω of right cosets of H in G is induced from the natural action of G. Another straightforward but important fact is that the statement $b_H(G) \leq k$ is equivalent to the existence of k subgroups of G conjugate to H in G such that their intersection is equal to H_G. Indeed, $H^{g_1} \cap \ldots \cap H^{g_k}$ is the pointwise stabiliser of $(Hg_1, \ldots, Hg_k) \in \Omega$. In particular, in the proofs of Theorems B1 and C1, we work with $\Gamma\mathrm{U}_n(q)$ and $\Gamma\mathrm{Sp}_n(q)$ rather than $\mathrm{P\Gamma U}_n(q)$ and $\mathrm{P\Gamma Sp}_n(q)$. To discuss unitary and symplectic groups simultaneously, we let $\Delta \in \{\mathrm{GU}_n(q), \mathrm{GSp}_n(q)\}$ and let $\mathbf{u}$ be 1 in symplectic case and 2 in unitary case, so $\Delta \in \mathrm{GL}_n(q^{\mathbf{u}})$.

Now we are ready to outline the main approaches and ideas of this paper. We use a combination of probabilistic, constructive and computational methods to establish our results. The probabilistic method is described in details in Section 2.3. It uses the ratio of points fixed by an element of the group $G \leq \mathrm{Sym}(\Omega)$ to obtain an upper bound on the probability $Q(G, c)$ that a randomly chosen c-tuple of points in Ω is not a base. By showing $Q(G, c) < 1$, one shows that there exists a base of size c. This method is used in most works related to base sizes of primitive permutation groups including [12, 13, 16, 17, 32, 34]. We use this method to obtain bounds for $b_S(S \cdot (\Delta \cap \mathrm{SL}_n(q^{\mathbf{u}})))$ for irreducible maximal solvable subgroups of Δ. In particular, with an explicit list of exceptions, we obtain $b_S(S \cdot (\Delta \cap \mathrm{SL}_n(q^{\mathbf{u}}))) \leq 3$ (see Theorems 3.21 and 3.22). Our results are refinements of [12, Theorem 1] in the sense that they provide better estimates for $b_H(G)$ for solvable H not lying in a $\mathcal{C}_1$-subgroup of an almost simple unitary or symplectic $G \leq \mathrm{PGL}_n(q^{\mathbf{u}})$.

The probabilistic method does not work for us in the general case since the fixed points ratio is much harder to estimate when S is reducible. Our reduction of the general case to the case of an irreducible subgroup of Δ is constructive. We illustrate this for a reducible maximal solvable subgroup S of Δ; the analysis for S not contained in $\mathrm{GL}_n(q^{\mathbf{u}})$ is much more technical and splits into number of smaller cases but applies the same general idea. Let S be a reducible maximal solvable subgroup of Δ. By Lemma 2.8, in a suitable basis, matrices in S have blocks on their diagonal and all the entries below these blocks are zero. The projection on each block forms an irreducible solvable matrix group of smaller dimension for which we can apply Theorems 2.11, 3.21 and 3.22. As a result, we obtain that the intersection of three (in most cases) conjugates of S consists of upper triangular matrices. Using the symmetry of matrices of shape (2.8), it is possible in most cases to adjust one of the conjugating elements so that the intersection of the three conjugates consists of diagonal matrices. Further, we explicitly construct a fourth element of $\Delta\mathrm{SL}_n(q^{\mathbf{u}})$ that, as a result of using it as a conjugating element, give us the subgroup of scalar matrices in the intersection of four conjugates of S, so $b_S(S \cdot (\Delta \cap \mathrm{SL}_n(q^{\mathbf{u}}))) \leq 4$. **Case 1** in **Step 2** of the proof of Theorem 4.5 illustrates well this method.

Both our theoretical tools, probabilistic method and the constructive method, turn out to be powerful, yet technical and demanding in terms of details. They require knowledge of the structure of solvable subgroups of classical groups. Here we heavily rely on the classical work of Suprunenko [37] and that of Manz and Wolf [33]. We also use some information from a recent work of Korhonen [31] where maximal solvable subgroups of $\mathrm{GL}_n(q)$, $\mathrm{GSp}_n(q)$ and $\mathrm{GO}_n^{\varepsilon}(q)$ are classified.

We use the computer algebra systems GAP [22] and MAGMA [6] to find and verify $b_S(G)$ for some small values of n and q when theoretical approach does not provide a sufficient result. Mostly, we use computations when an irreducible maximal solvable subgroup of Δ is "too large" and it is much easier to obtain results this way rather than establish a suitable estimate for $Q(G, c)$. For details see [4, Section 2.7].

The paper is organised as follows. In Section 2 we present necessary definitions and preliminary results. In Section 3 we obtain bounds for $b_S(S \cdot (\Delta \cap \mathrm{SL}_n(q^{\mathbf{u}}))$ for irreducible maximal solvable subgroups S of $\Delta \in \{\mathrm{GU}_n(q), \mathrm{GSp}_n(q)\}$. Finally, we use these results to prove Theorems B1, C1 and C2 in Sections 4 and 5.

2. Definitions and preliminaries

All group actions we use are right actions. For example, the action of a linear transformation g of a vector space V on $v \in V$ is $(v)g \in V$.

Let p be a prime and $q = p^f$, $f \in \mathbb{N}$. Denote a finite field of size q by $\mathbb{F}_q$, its algebraic closure by $\overline{\mathbb{F}_q}$ and the multiplicative group of $\mathbb{F}_q$ by $\mathbb{F}_q^*$. Throughout, unless stated otherwise, $V = \mathbb{F}_{q^{\mathbf{u}}}^n$ denotes a vector space of dimension n over $\mathbb{F}_{q^{\mathbf{u}}}$ with $\mathbf{u} \in \{1, 2\}$.

For finite classical groups we follow the notation of [30]. For algebraic groups our standard references are [20, Chapter 1], [25, Chapter 1] and [26].

We reserve the letter β for a **basis** of V. A basis is an ordered set. If $\alpha \in \mathrm{Aut}(\mathbb{F})$, then $\phi_\beta(\alpha)$ denotes the unique $g \in \Gamma\mathrm{L}(V, \mathbb{F})$ such that

$$\left(\sum_{i=1}^{n} \lambda_i v_i\right) \phi_\beta(\alpha) = \sum_{i=1}^{n} \lambda_i^{\alpha} v_i \tag{2.1}$$

where $\beta = \{v_1, \ldots, v_n\}$. If $\mathbb{F} = \mathbb{F}_{q^{\mathbf{u}}}$ and $\alpha \in \mathrm{Aut}(\mathbb{F})$ is such that $\lambda^\alpha = \lambda^p$ for all $\lambda \in \mathbb{F}$, then we denote $\phi_\beta(\alpha)$ by ϕ_β or simply ϕ when β is understood. It is routine to check (see [30, §2.2]) that

$$\Gamma\mathrm{L}(V, \mathbb{F}_{q^{\mathbf{u}}}) = \mathrm{GL}(V, \mathbb{F}_{q^{\mathbf{u}}}) \rtimes \langle \phi \rangle \cong \mathrm{GL}_n(q^{\mathbf{u}}) \rtimes \langle \phi \rangle.$$

We fix the following notation.

$\mathbf{F}(G)$	**Fitting subgroup** of a finite group G (unique maximal normal nilpotent subgroup);
$O_\pi(G)$	unique maximal normal π-subgroup for a set of primes π;
$Z(G)$	**center** of a group G;
g^G	conjugacy class of $g \in G$;
$A \rtimes B$	semidirect product of groups A and B with A normal;

$\mathrm{Sym}(n)$	symmetric group of degree n;
$\mathrm{sgn}(\pi)$	**sign** of a permutation π;
$M_n(\mathbb{F})$	algebra of all $n \times n$ matrices over $\mathbb{F}$;
$\mathrm{diag}(\alpha_1, \dots, \alpha_n)$	**diagonal** matrix with entries $\alpha_1, \dots, \alpha_n$ on its diagonal;
$\mathrm{diag}[g_1, \dots, g_k]$	**block-diagonal** matrix with blocks $g_1, \dots, g_k$ on its diagonal;
$\mathrm{perm}(\sigma)$	**permutation matrix** corresponding to $\sigma \in \mathrm{Sym}(n)$;
$g^\top$	**transpose** of a matrix g;
$\mathrm{Det}(H)$	$\{\det(h) \mid h \in H\}$ for $H \leq \mathrm{GL}(V)$;
$g \otimes h$	**Kronecker product** $\begin{pmatrix} g \cdot h_{1,1} & \dots & g \cdot h_{1,m} \\ \dots & \dots & \dots \\ g \cdot h_{m,1} & \dots & g \cdot h_{m,m} \end{pmatrix} \in \mathrm{GL}_{nm}(q)$ for $g \in \mathrm{GL}_n(q)$ and $h \in \mathrm{GL}_m(q)$;
$D(G)$	subgroup of all diagonal matrices of a matrix group G;
$RT(G)$	subgroup of all upper-triangular matrices of a matrix group G;
p'	set of all primes except p;
(a, b)	greatest common divisor of integers a and b.

It is convenient to view the symmetric group as a group of permutation matrices. We define the **wreath product** of $X \leq \mathrm{GL}_n(q)$ and a group of permutation matrices $Y \leq \mathrm{GL}_m(q)$ as the matrix group $X \wr Y \leq \mathrm{GL}_{nm}(q)$ obtained by replacing the entries 1 and 0 in every matrix in Y by arbitrary matrices in X and by zero $(n \times n)$ matrices respectively.

Let A be an $(nm \times nm)$ matrix. We can view A as the matrix

$$\begin{pmatrix} A_{11} & \dots & A_{1m} \\ \dots & \dots & \dots \\ A_{m1} & \dots & A_{mm} \end{pmatrix}$$

where the A_{ij} are $(n \times n)$ matrices. The vector $(A_{i1}, \dots, A_{im})$ is the i-th $(n \times n)$-**row** of A.

2.1. **Classical forms and groups.** Let us recall some definitions and results from [30, Chapter 1].

Let $\mathbf{f}$ be a non-degenerate unitary or symplectic form, so $\mathbf{u}$ is 2 or 1 respectively. If $\mathbf{f}$ is fixed, then we write (v, w) instead of $\mathbf{f}(v, w)$ for convenience.

The pair $(V, \mathbf{f})$ is a **unitary (symplectic) space**. Two unitary (symplectic) spaces $(V_1, \mathbf{f}_1)$ and $(V_2, \mathbf{f}_2)$ are **isometric** if there exists an isomorphism of vector spaces $\varphi : V_1 \to V_2$ such that

$$\mathbf{f}_1(v, u) = \mathbf{f}_2((v)\varphi, (u)\varphi)$$

for every v and u from V_1. Such φ is an **isometry.** A **similarity** of unitary (symplectic) spaces $(V_1, \mathbf{f}_1)$ and $(V_2, \mathbf{f}_2)$ is an isomorphism of vector spaces $\varphi : V_1 \to V_2$ such that there exists $\lambda \in \mathbb{F}_{q^{\mathbf{u}}}$ with

$$\mathbf{f}_1(v, u) = \lambda \mathbf{f}_2((v)\varphi, (u)\varphi) \tag{2.2}$$

for every v and u from V_1.

Let us fix $\mathbf{f}$ to be a non-degenerate unitary or symplectic form on V for the rest of the section. Let W be a subspace of V. If the restriction $\mathbf{f}_W$ of $\mathbf{f}$ to W is non-degenerate, then W is a **non-degenerate subspace** of V. If $\mathbf{f}_W = 0$, then W is a **totally isotropic subspace** of V.

Two subspaces U and W of V are **orthogonal** if $(u, w) = 0$ for all $u \in U$ and all $w \in W$. We write $U \perp W$ for the direct sum of orthogonal subspaces. The **orthogonal complement** $W^\perp$

of W in V is
$$\{v \in V \mid (v,u) = 0 \text{ for all } u \in W\}.$$

Let $I(V,\mathbf{f})$ and $\Delta(V,\mathbf{f})$ be the group of all $\mathbf{f}$-isometries and all $\mathbf{f}$-similarities from V to itself respectively. By definition, $I(V,\mathbf{f})$ and $\Delta(V,\mathbf{f})$ are subgroups of $\mathrm{GL}(V)$, so $\Sigma(V,\mathbf{f}) := \mathrm{SL}(V) \cap I(V,\mathbf{f})$ is well-defined.

All non-degenerate unitary (respectively symplectic) spaces of the same dimension over $\mathbb{F}_{q^{\mathbf{u}}}$ are isometric by the following lemmas. Here δ_{ij} is the Kronecker delta.

Lemma 2.1 ([30, Propositions 2.3.1 and 2.3.2]). *Let* $\mathbf{f}$ *be unitary.*

(1) *The space* $(V,\mathbf{f})$ *has an orthonormal basis.*
(2) *The space* $(V,\mathbf{f})$ *has a basis*
$$\begin{cases} \{f_1,\dots,f_m,e_1,\dots,e_m\}, & \text{if } n = 2m \\ \{f_1,\dots,f_m,x,e_1,\dots,e_m\}, & \text{if } n = 2m+1 \end{cases} \tag{2.3}$$
where $(e_i,e_j) = (f_i,f_j) = 0$, $(e_i,f_j) = \delta_{ij}$ *and* $(e_i,x) = (f_i,x) = 0$ *for all* i,j, *and* $(x,x) = 1$.

Lemma 2.2 ([30, Proposition 2.4.1]). *Let* $\mathbf{f}$ *be symplectic. The dimension* n *of* V *is even and the space* $(V,\mathbf{f})$ *has a basis*
$$\{f_1,\dots,f_m,e_1,\dots,e_m\}, \tag{2.4}$$
where $2m = n$, $(e_i,e_j) = (f_i,f_j) = 0$ *and* $(e_i,f_j) = \delta_{ij}$ *for all* i,j.

Hence, for a non-degenerate unitary or symplectic space $(V,\mathbf{f})$, the groups $\Sigma(V,\mathbf{f})$, $I(V,\mathbf{f})$ and $\Delta(V,\mathbf{f})$ are also defined uniquely (up to conjugation in $\mathrm{GL}(V)$) by $\dim V$ and q.

An **f-semisimilarity** is $g \in \Gamma\mathrm{L}_n(q^{\mathbf{u}})$ such that there exist $\lambda \in \mathbb{F}_{q^{\mathbf{u}}}^*$ and $\alpha \in \mathrm{Aut}(\mathbb{F}_{q^{\mathbf{u}}})$ satisfying
$$\mathbf{f}(vg,ug) = \lambda \mathbf{f}(v,u)^{\alpha} \text{ for all } v,u \in V. \tag{2.5}$$
By [30, Lemma 2.1.2], α is determined uniquely by g, and $\alpha = \sigma(g)$. We denote the group of $\mathbf{f}$-semisimilarities of V by $\Gamma(V,\mathbf{f})$. It is easy to see that
$$\Delta(V,\mathbf{f}) \leq \Gamma(V,\mathbf{f}).$$

Definition 2.3. By [30, Lemma 2.1.2], if $\mathbf{f}$ is non-degenerate, then the λ in (2.2) and (2.5) are uniquely determined by g. Moreover, there exists a homomorphism $\tau : \Delta(V,\mathbf{f}) \to \mathbb{F}_{q^{\mathbf{u}}}^*$ satisfying $\tau(g) = \lambda$.

We say that we work on the case **U** or **S** when $\mathbf{f}$ is unitary or symplectic respectively. Notice that $\mathbf{u} = 2$ in the case **U** and $\mathbf{u} = 1$ otherwise. We summarise notation for the groups Σ, I, Δ and Γ in Table 2. For more details on classical groups and the equalities claimed in the table see [30, §2.1].

Denote the identity $(n \times n)$ matrix by I_n and let J_{2k} be the matrix
$$\begin{pmatrix} & I_k \\ -I_k & \end{pmatrix}.$$
For $g \in \mathrm{GL}_n(q^{\mathbf{u}})$ let $\overline{g}$ be the matrix obtained from g by taking every entry to the q-th power (so if $\mathbf{u} = 1$, then $\overline{g} = g$). We write $g^{\dagger}$ for $(\overline{g}^{\top})^{-1}$ and $X^{\dagger}$ for $\{g^{\dagger} \mid g \in X\}$, where $X \subseteq \mathrm{GL}_n(q^{\mathbf{u}})$.

Fix a basis $\beta = \{v_1,\dots,v_n\}$ of V and denote by $\mathbf{f}_\beta$ the matrix whose (i,j) entry is $\mathbf{f}(v_i,v_j)$. By fixing the basis, we identify $I(V,\mathbf{f})$ and $\Delta(V,\mathbf{f})$ with the matrix groups
$$\{g \in \mathrm{GL}_n(q^{\mathbf{u}}) \mid g\mathbf{f}_\beta\overline{g}^{\top} = \mathbf{f}_\beta\} \text{ and } \{g \in \mathrm{GL}_n(q^{\mathbf{u}}) \mid g\mathbf{f}_\beta\overline{g}^{\top} = \lambda\mathbf{f}_\beta, \lambda \in \mathbb{F}_{q^{\mathbf{u}}}^*\} \tag{2.6}$$

Table 2. Notation for classical groups

case		notation	terminology
U	Σ	$\mathrm{SU}(V)$	unitary groups
	I	$\mathrm{GU}(V)$	
	Γ	$\Gamma\mathrm{U}(V)$	
S	$\Sigma = I$	$\mathrm{Sp}(V)$	symplectic groups
	Δ	$\mathrm{GSp}(V)$	
	Γ	$\Gamma\mathrm{Sp}(V)$	

respectively; we identify $\Gamma(V,\mathbf{f})$ with the subgroup $\Gamma(V,\mathbf{f})_\beta \leq \Gamma\mathrm{L}(V,\beta)$ of $\mathbf{f}$-semisimilarities.

Denote the group of matrices representing the isometries from $I(V,\mathbf{f})$ with respect to a basis β such that $\mathbf{f}_\beta = \Phi$ by $\mathrm{GU}_n(q,\Phi)$ (respectively $\mathrm{Sp}_n(q,\Phi)$) or $\mathrm{GU}_n(q,\beta)$ (respectively $\mathrm{Sp}_n(q,\beta)$). We write $\mathrm{GU}_n(q)$ (respectively $\mathrm{Sp}_n(q)$) instead of $\mathrm{GU}_n(q,I_n)$ (respectively $\mathrm{Sp}_n(q,J_n)$) for simplicity; we use similar notation for $\Sigma(V,\mathbf{f})$, $\Delta(V,\mathbf{f})$ and $\Gamma(V,\mathbf{f})$ in cases **U** and **S**. We also use $\mathrm{GL}_n^\varepsilon(q)$ with $\varepsilon \in \{+,-\}$ where $\mathrm{GL}_n^+(q) = \mathrm{GL}_n(q)$ and $\mathrm{GL}_n^-(q) = \mathrm{GU}_n(q)$.

Note the following observations and notation:

- In some literature $\Delta(V,\mathbf{f})$ for case **S** is denoted by $\mathrm{CSp}(V)$ and called the "conformal symplectic group".
- If β is as in Lemmas 2.1 and 2.2 for cases **U** and **S** respectively, then $\phi_\beta \in \Gamma(V,\mathbf{f})$ and $\Gamma(V,\mathbf{f})_\beta = \Delta(V,\mathbf{f})_\beta \rtimes \langle\phi_\beta\rangle$.
- The group $\Delta(V,\mathbf{f})$ for case **U** is omitted in Table 2 since
$$\Delta(V,\mathbf{f}) = I(V,\mathbf{f}) \cdot \mathbb{F}_q^*.$$
Therefore, $\Delta(V,\mathbf{f})_\beta \rtimes \langle\phi_\beta\rangle$ and $I(V,\mathbf{f})_\beta \rtimes \langle\phi_\beta\rangle$ (and their maximal solvable subgroups) coincide modulo scalars. It is more convenient for us to work with $I(V,\mathbf{f})_\beta \rtimes \langle\phi_\beta\rangle$, so in what follows we abuse notation by letting
$$\Gamma(V,\mathbf{f}) = I(V,\mathbf{f}) \rtimes \langle\phi_\beta\rangle$$
for an orthonormal basis β in case **U**.
- If $\Sigma(V,\mathbf{f}) \leq G \leq \Gamma(V,\mathbf{f})$, then G is solvable if and only if $\Sigma(V,\mathbf{f})$ is solvable since $\Delta(V,\mathbf{f})/\Sigma(V,\mathbf{f})$ and $\Gamma(V,\mathbf{f})/\Delta(V,\mathbf{f})$ are abelian. Therefore, such G is solvable if and only if either $n=1$ or $\Sigma(V,\mathbf{f})$ is one of the following groups: $\mathrm{SL}_2(q) = \mathrm{Sp}_2(q) \cong \mathrm{SU}_2(q)$ for $q \in \{2,3\}$, $\mathrm{SU}_3(2)$. We often write "G is not solvable" where we ignore these groups.

We state a particular case of Witt's Lemma, which we use later. For a proof see [1, §20].

Lemma 2.4. *Assume that $(V_1,\mathbf{f}_1)$, $(V_2,\mathbf{f}_2)$ are isometric unitary (symplectic) spaces and W_i is a subspace of V_i for $i=1,2$. If there is an isometry g from $(W_1,\mathbf{f}_1)$ to $(W_2,\mathbf{f}_2)$, then g extends to an isometry from $(V_1,\mathbf{f}_1)$ to $(V_2,\mathbf{f}_2)$.*

2.2. **Miscellaneous results.** The following lemma is, in some sense, a generalisation of the well known Clifford's Theorem to semilinear groups.

Lemma 2.5. *Let G be a subgroup of $\Gamma\mathrm{L}_n(q)$ stabilising no non-zero proper subspaces of $V = \mathbb{F}_q^n$ and let $M = G \cap \mathrm{GL}_n(q)$. Then M is completely reducible and stabilises a decomposition*
$$V = V_1 \oplus \ldots \oplus V_k;\ k \geq 1$$
where each V_i is $\mathbb{F}_q[M]$-irreducible and G/M permutes the V_i cyclically. Moreover, every $\mathbb{F}_q$-irreducible M-invariant subspace of V has dimension $m = n/k$.

Proof. If M is irreducible, the lemma is trivial, so let us assume that M is reducible.

Let V_1 be an irreducible $\mathbb{F}_q[M]$-submodule of V and let $m = \dim V_1$. Since $M \trianglelefteq G$, $V_1\varphi$ is an irreducible $\mathbb{F}_q[M]$-submodule of V of dimension m for all $\varphi \in G$. Let $\varphi \in G$ be such that $M\varphi$ is a generator of G/M, so $|\varphi| = |S : M| = r$. Let $V_i = V_1\varphi^{i-1}$ for $i \in \{1, \dots, r\}$, so $V = \sum_{i=1}^r V_i$ since there is no non-zero proper G-invariant subspaces of V.

We claim that

$$V = V_1 \oplus \ldots \oplus V_{n/m},$$

so M is completely reducible. The proof of this fact is similar to a proof of Clifford's Theorem. Let $W_i = \sum_{j=1}^i V_i$. If $W_{i+1} = W_i$ for some i, then W_i is G-invariant. Indeed,

$$(W_i)\varphi = (V_1\varphi + \ldots + V_i\varphi) = (V_2 + \ldots + V_{i+1}) \subseteq W_{i+1} = W_i$$

and $(W_i)\varphi = W_i$ since $\dim(W_i)\varphi = \dim W_i$. On the other hand, if $W_{i+1} > W_i$, then $W_i \cap U_{i+1}$ is a proper $\mathbb{F}_q[M]$-submodule of U_{i+1}, so it must be zero. Hence $W_{i+1} = W_i \oplus V_{i+1}$. Since $k = 1$, $W_{i+1} > W_i$ for $1 \leq i \leq n/m - 1$. So, by induction,

$$V = W_{n/m} = V_1 \oplus \ldots \oplus V_{n/m}$$

and M is completely reducible. In particular, since each V_i is $\mathbb{F}_q$-irreducible, every $\mathbb{F}_q$-irreducible M-invariant subspace of V has dimension m. □

The next three lemmas provide information on $\Gamma(V, \mathbf{f})$ and its subgroups. Here $\mathbf{f}$ is unitary or symplectic; by default, we assume that such a form $\mathbf{f}$ is non-degenerate.

Lemma 2.6 ([1, (5.5)]). *Let $H \leq \Gamma(V, \mathbf{f})$, with $\mathbf{f}$ unitary or symplectic, be irreducible. Let L be a non-scalar normal subgroup of H contained in $\mathrm{GL}(V)$. Let $\{V_i \mid 1 \leq i \leq k\}$ be the homogeneous components of L on V and assume $k > 1$. One of the following holds:*

(1)
$$V = \perp_{1 \leq i \leq k} V_i$$
with V_i non-degenerate and isometric to V_j for each $1 \leq i \leq j \leq k$;

(2)
$$V = \perp_{1 \leq i \leq k/2} U_i$$
with $U_i = V_{2i-1} \oplus V_{2i}$ where U_i is non-degenerate and isometric to U_j for $1 \leq i \leq j \leq k/2$, and V_i is totally isotropic for each $1 \leq i \leq k$.

Lemma 2.7. *Let $\mathbf{f}$ be a non-degenerate unitary or symplectic form on V. If β is a basis of V such that $\mathbf{f}_\beta^{\phi_\beta} = \mathbf{f}_\beta$, then $\Gamma(V, \mathbf{f})_\beta = \Delta(V, \mathbf{f})_\beta \rtimes \langle \phi_\beta \rangle$.*

Proof. Clearly, $\Delta(V, \mathbf{f})_\beta \cap \langle \phi_\beta \rangle = 1$, so it suffices to show that ϕ_β normalises $\Delta(V, \mathbf{f})_\beta$ and is a semisimilarity of $(V, \mathbf{f})$. Let $g \in \Delta(V, \mathbf{f})_\beta$, so

$$g\mathbf{f}_\beta \overline{g}^\top = \lambda \mathbf{f}_\beta$$

for some $\lambda \in \mathbb{F}_q^*$. Therefore,

$$g^{\phi_\beta}\mathbf{f}_\beta \overline{(g^{\phi_\beta})}^\top = g^{\phi_\beta}\mathbf{f}_\beta^{\phi_\beta} \overline{(g^{\phi_\beta})}^\top = (g\mathbf{f}_\beta\overline{g}^\top)^{\phi_\beta} = (\lambda \mathbf{f}_\beta)^{\phi_\beta} = \lambda^p \mathbf{f}_\beta,$$

and $g^{\phi_\beta} \in \Delta(V, \mathbf{f})_\beta$.

Let $v, u \in V$ have coefficients $(\alpha_1, \dots, \alpha_n)$ and $(\delta_1, \dots, \delta_n)$ with respect to β respectively. Therefore,

$$\begin{aligned}(v\phi_\beta, u\phi_\beta) &= (\alpha_1^p, \dots, \alpha_n^p)\mathbf{f}_\beta(\overline{\delta_1}^p, \dots, \overline{\delta_n}^p)^\top \\ &= (\alpha_1^p, \dots, \alpha_n^p)\mathbf{f}_\beta^{\phi_\beta}(\overline{\delta_1}^p, \dots, \overline{\delta_n}^p)^\top \\ &= (v, u)^p,\end{aligned}$$

so ϕ_β is a semisimilarity. □

Lemma 2.8. *Recall that $q = p^f$. Let $H \leq \Gamma(V, \mathbf{f})$ with $\mathbf{f}$ unitary or symplectic. There exists a basis β such that $\mathbf{f}_\beta$ is*

$$\begin{pmatrix} & & & & & & & I_{n_1} \\ & & & & & & \iddots & \\ & & & & & I_{n_k} & & \\ & & & I_{n_{k+1}} & & & & \\ & & & & \ddots & & & \\ & & & & & I_{n_{k+l}} & & \\ & & I_{n_k} & & & & & \\ & \iddots & & & & & & \\ I_{n_1} & & & & & & & \end{pmatrix} \text{ or } \begin{pmatrix} & & & & & & & I_{n_1} \\ & & & & & & \iddots & \\ & & & & & I_{n_k} & & \\ & & & J_{n_{k+1}} & & & & \\ & & & & \ddots & & & \\ & & & & & J_{n_{k+l}} & & \\ & & -I_{n_k} & & & & & \\ & \iddots & & & & & & \\ -I_{n_1} & & & & & & & \end{pmatrix} \tag{2.7}$$

in cases $\mathbf{U}$ and $\mathbf{S}$ respectively. Moreover, if $\varphi \in H_\beta$, then $\varphi = (\phi_\beta)^j g$ with

$$j \in \{1, \dots, \mathbf{u}f - 1\}$$

and

$$g = \left(\begin{array}{ccc|ccc|ccc} \tau(g)\gamma_1(g)^\dagger & * & * & * & \dots & * & * & * & * \\ & \ddots & & & & \ddots & & & \\ 0 & & \tau(g)\gamma_k(g)^\dagger & * & * & * & \dots & * & * \\ \hline & & & \gamma_{k+1}(g) & & 0 & * & \dots & * \\ & & & & \ddots & & & \ddots & \\ & & & 0 & & \gamma_{k+l}(g) & * & \dots & * \\ \hline & & & & & & \gamma_k(g) & * & * \\ & & & & & & & \ddots & * \\ 0 & & & & & & 0 & & \gamma_1(g) \end{array}\right) \tag{2.8}$$

where $\tau(g)$ is as in Definition 2.3, γ_i is a homomorphism from H to $\Gamma\mathrm{L}_{n_i}(q^{\mathbf{u}})$ if $i \leq k$, and from H to $\Gamma\mathrm{U}_{n_i}(q)$ or $\Gamma\mathrm{Sp}_{n_i}(q)$, in cases $\mathbf{U}$ and $\mathbf{S}$ respectively, if $i > k$. Furthermore, $\gamma_i(H)$ is an irreducible subgroup of $\Gamma\mathrm{L}_{n_i}(q^{\mathbf{u}})$ for every i and $\gamma_i(H \cap \mathrm{GL}_n(q^{\mathbf{u}})) \leq \mathrm{GL}_{n_i}(q^{\mathbf{u}})$.

Proof. If H is an irreducible subgroup of $\Gamma\mathrm{L}(V, \mathbb{F}_{q^{\mathbf{u}}})$, then by Lemmas 2.1 and 2.2 we can take $\mathbf{f}_\beta$ to be I_n or J_n in cases $\mathbf{U}$ and $\mathbf{S}$, and there is nothing to prove. So assume that there is a proper H-invariant subspace W of $V = \mathbb{F}_{q^{\mathbf{u}}}^n$ on which H acts irreducibly. Therefore, W is either non-degenerate or totally isotropic. If V has no totally isotropic H-invariant subspace, then V is the direct sum of pairwise orthogonal H-invariant non-degenerate subspaces, so $k = 0$ and the lemma follows.

Assume that W is totally isotropic. By Lemma 2.4 we can assume that V has a basis β as in (2.3) such that

$$W = \langle e_{(n-n_1+1)}, \dots, e_n \rangle$$

where $n_1 = \dim W$. Let U be the subspace spanned by

$$\beta \backslash \{f_{(n-n_1+1)}, \dots, f_n, e_{(n-n_1+1)}, \dots, e_n\}.$$

Notice that U is non-degenerate. Let $\beta_2 := \{v_1, \dots, v_{n-2n_1}\}$ be a basis of U such that β_2 is orthonormal in case **U** and as in (2.4) in case **S**.

Let us define a basis

$$\beta_1 := \{f_{(n-n_1+1)}, \dots, f_n, v_1, \dots, v_{n-2n_1}, e_{(n-n_1+1)}, \dots, e_n\}.$$

Hence

$$\mathbf{f}_{\beta_1} = \begin{pmatrix} & & I_{n_1} \\ & \Phi & \\ (-1)^{\mathbf{u}} I_{n_1} & & \end{pmatrix}$$

with Φ equal to I_{n-2n_1} and J_{n-2n_1} in cases **U** and **S** respectively. Since H stabilises W, it also stabilises $W^\perp = \langle v_1, \dots, v_{n-2n_1}, e_{(n-n_1+1)}, \dots, e_n \rangle$. By Lemma 2.7, if $\varphi \in H_{\beta_1}$, then $\varphi = (\phi_{\beta_1})^j g$ with $g \in \mathrm{GU}_n(q, \mathbf{f}_{\beta_1})$ or $\mathrm{GSp}_n(q, \mathbf{f}_{\beta_1})$ respectively, so, by (2.6),

$$g = \begin{pmatrix} \tau(g)(g_W)^\dagger & * & * \\ 0 & g_1 & * \\ 0 & 0 & g_W \end{pmatrix},$$

where g_1 is an $(n - 2n_1) \times (n - 2n_1)$ matrix. If $n = 2n_1$, then the lemma follows. We proceed by induction on $n - 2n_1$ using the case $n - 2n_1 = 0$ as the base.

Assume that $n > 2n_1$. Since $g\mathbf{f}_{\beta_1}\overline{g}^\top = \tau(g)\mathbf{f}_{\beta_1}$,

$$g_1 \Phi \overline{g_1}^\top = \tau(g)\Phi.$$

Thus, g_1 is a similarity of

$$(\langle v_1, \dots, v_{n-2n_1} \rangle, \mathbf{f_1}),$$

and

$$(\phi_{\beta_2})^j g_1 \in \Gamma(\langle v_1, \dots, v_{n-2n_1} \rangle, \mathbf{f_1})$$

where $\mathbf{f_1}$ is the restriction of $\mathbf{f}$ to $\langle v_1, \dots, v_{n-2n_1} \rangle$. Notice that $(\phi_{\beta_2})^j g_1$ is the restriction of φ to $W^\perp/W$. So there exists a homomorphism ψ from H_{β_1} to $\Gamma\mathrm{U}_{n-2n_1}(q)$ in case **U** and $\Gamma\mathrm{Sp}_{n-2n_1}(q)$ in case **S** defined by $\psi : g \mapsto (\phi_{\beta_2})^j g_1$. Applying induction to $\psi(H_{\beta_1})$, we obtain the lemma. □

The following lemma plays an important role in our proof of Theorems B1 and C1.

Lemma 2.9. *Let $\Gamma \in \{\Gamma\mathrm{U}_n(q), \Gamma\mathrm{Sp}_n(q)\}$. Let $n \geq 2$ and let q be such that Γ is not solvable. Let β be a basis of V such that $\mathbf{f}_\beta^{\phi_\beta} = \mathbf{f}_\beta$ and let $\phi = \phi_\beta$. If $H \leq \Gamma$ and $H \cap \mathrm{GL}_n(q^{\mathbf{u}})$ consists of scalar matrices, then there exists $b \in \Gamma \cap \mathrm{GL}_n(q^{\mathbf{u}})$ such that every element of H^b has shape $\phi^i g$ for some $i \in \{1, \dots, \mathbf{u}f\}$ and $g \in Z(\mathrm{GL}_n(q^{\mathbf{u}}))$.*

Proof. Let $Z = Z(\mathrm{GL}_n(q^{\mathbf{u}}) \cap \Gamma)$. Notice that Γ/Z is almost simple. Let G_0 and $\hat{G}$ be the socle of Γ/Z and the group of inner-diagonal automorphisms of G_0 respectively. Therefore, $\hat{G} = (\Gamma \cap \mathrm{GL}_n(q^{\mathbf{u}}))/Z$. Without loss of generality, we may assume $Z \leq H$. Observe $H \cap \mathrm{GL}_n(q^{\mathbf{u}}) = Z$, so H/Z is cyclic and consists of field automorphisms of G_0. Let $\varphi \in H$ be such that $\langle Z\varphi \rangle = H/Z$. By Lemma 2.7,

$$\Gamma = (\Gamma \cap \mathrm{GL}_n(q^{\mathbf{u}})) \rtimes \langle \phi \rangle,$$

so $\varphi \in \phi^i(\Gamma \cap \mathrm{GL}_n(q^{\mathbf{u}}))$ for some $i \in \{1, \dots, \mathbf{u}f\}$ and $Z\varphi \in (Z\phi^i)\hat{G}$.

By [24, (7-2)], $Z\varphi$ and $Z\phi^i$ are conjugate in $\hat{G}$, so there exists $Zb \in \Gamma/Z \cap \mathrm{PGL}_n(q^{\mathbf{u}})$ such that $(Z\varphi)^{Zb} = Z\phi^i$ for some $i \in \{1, \dots, \mathbf{u}f\}$. Therefore, $H^b = Z\langle \varphi^b \rangle = Z\langle \phi^i \rangle = \langle \phi^i \rangle Z$. □

Further, we collect results on intersections of subgroups and base sizes.

Theorem 2.10 ([39])**.** *If A and B are abelian subgroups of a finite group G, then there exists $x \in G$ such that $A \cap B^x \leq \mathbf{F}(G)$.*

The following two results are the main result of [5, Section 3] and its corollary proved in [5, Section 4].

Theorem 2.11. *Let S be an irreducible maximal solvable subgroup of $\mathrm{GL}_n(q)$ with $n \geq 2$, and (n, q) is neither $(2, 2)$ nor $(2, 3)$. Then either $b_S(S \cdot \mathrm{SL}_n(q)) = 2$ or one of the following holds:*

1. *$n = 2$, $q > 3$ is odd, S is the normaliser of a Singer cycle and $b_S(S \cdot \mathrm{SL}_2(q)) = 3$. If $q > 5$, then there exists $x \in \mathrm{SL}_2(q)$ such that $S \cap S^x \leq D(\mathrm{GL}_2(q))$;*
2. *$n = 2$, $q \geq 4$ is even, S is the normaliser of a Singer cycle and $b_S(S \cdot \mathrm{SL}_2(q)) = 3$. In this case there exists $x \in \mathrm{SL}_2(q)$ such that $S \cap S^x \leq RT(\mathrm{GL}_2(q))$;*
3. *$n = 2$, $q = 9$, S (up to conjugacy) is generated by all matrices with entries in $\mathbb{F}_3$ and scalar matrices, so $S = \mathrm{GL}_2(3) \cdot Z(\mathrm{GL}_2(9))$, and $b_S(S \cdot \mathrm{SL}_2(9)) = 3$. In this case there exists $x \in \mathrm{SL}_2(9)$ such that $S \cap S^x \leq RT(\mathrm{GL}_2(9))$;*
4. *$n = 2$, $q \in \{5, 7\}$, and S is an absolutely irreducible subgroup such that $S/Z(\mathrm{GL}_2(q))$ is isomorphic to $2^2.\mathrm{Sp}_2(2)$. Here $b_S(S \cdot \mathrm{SL}_2(q))$ is 4 and 3 for q equal to 5 and 7 respectively;*
5. *$n = 3$, $q = 2$, S is the normaliser of a Singer cycle and $b_S(S \cdot \mathrm{SL}_3(2)) = 3$;*
6. *$n = 4$, $q = 3$, $S = \mathrm{GL}_2(3) \wr \mathrm{Sym}(2)$ and $b_S(S \cdot \mathrm{SL}_4(3)) = 3$. In this case there exists $x \in \mathrm{SL}_4(3)$ such that $S \cap S^x \leq RT(\mathrm{GL}_4(3))$.*

Lemma 2.12. *If $k = 1$, then either $S \cap \mathrm{GL}_n(q)$ is an irreducible solvable subgroup of $\mathrm{GL}_n(q)$ or there exists $x \in \mathrm{SL}_n(q)$ such that $S \cap S^x \cap \mathrm{GL}_n(q) \leq Z(\mathrm{GL}_n(q))$.*

Lemma 2.13. *Let H be a maximal solvable subgroup of $\mathrm{PGL}_2(q)$. Then $b_H(H \cdot \mathrm{PSL}_2(q)) \leq 3$ unless $q = 5$, H is the image of S in Theorem 2.11(4), and $b_H(H \cdot \mathrm{PSL}_2(5)) = 4$.*

Proof. Let S be the full preimage of G in $\mathrm{GL}_2(q)$, so $b_H(H \cdot \mathrm{PSL}_2(q)) = b_S(S \cdot \mathrm{SL}_n(q))$. If S is irreducible, then the lemma follows from Theorem 2.11. If S is reducible, then, in suitable basis, S is the group P_1 of upper-triangular matrices. Now the lemma follows by [14, Proposition 4.1]. □

We conclude the section with three technical lemmas.

Lemma 2.14. *For every prime power $q = p^f$ there exists $\alpha \in \mathbb{F}_{q^2}$ such that $\alpha + \alpha^q = 1$.*

Proof. If $p \neq 2$, then $2^{-1} \in \mathbb{F}_q^*$, so $2^{-1} + (2^{-1})^q = 2^{-1} + 2^{-1} = 1$.

Let $p = 2$. Let $y \in \overline{\mathbb{F}_2}$ be a root of polynomial $x^q + x + 1 = 0$. Hence

$$y^{q^2} = (y^q)^q = (y+1)^q = y^q + 1 = y + 1 + 1 = y,$$

so $y \in \mathbb{F}_{q^2}$. $\square$

Lemma 2.15. *Let η be a generator of $\mathbb{F}_{q^2}^*$ and let $\theta = \eta^{q-1}$. If $\theta^{p^j-1} = 1$ for some $j \in \{0, 1, \dots, 2f-1\}$, then $j = 0$.*

Proof. Notice that $|\theta| = p^f + 1$. Let $j \in \{1, \dots, 2f\}$ be minimal such that $p^f + 1$ divides $p^j - 1$. Hence $p^f + 1$ divides $(p^{2f} - 1, p^j - 1) = p^{(2f,j)} - 1$. Therefore, $(2f, j) > f$, so $j = 2f$. $\square$

The following technical lemma is proved in [5, Section 3].

Lemma 2.16. *Let $H \leq X \wr Y$, where $X \leq \mathrm{GL}_m(q)$, $Y \leq \mathrm{Sym}(k)$. Let $A(k) = (y_{ij}) \in GL_k(q)$ be the inverse of the Jordan block $J_{1,k}$ and let x_i for $i = 1, \dots, k$ be arbitrary elements of X. Define $x \in \mathrm{GL}_{mk}(q)$ to be*

$$\mathrm{diag}(x_1, \dots, x_k)(I_m \otimes A(k)) = \begin{pmatrix} y_{11}x_1 & y_{12}x_1 & \dots & y_{1k}x_1 \\ y_{21}x_2 & y_{22}x_2 & \dots & y_{2k}x_2 \\ \vdots & & & \vdots \\ y_{k1}x_k & y_{k2}x_k & \dots & y_{kk}x_k \end{pmatrix},$$

Let $h = \mathrm{diag}[D_1, \dots, D_k] \cdot s \in H$, where $D_i \in X$ and $s \in Y$, so h is obtained from the permutation matrix s by replacing 1 in the j-th row by the $(m \times m)$ matrix D_j for $j = 1, \dots, k$ and replacing each zero by an $(m \times m)$ zero matrix. If $h^x \in H$, then s is trivial and $D_j^{x_j} = D_{j+1}^{x_{j+1}}$ for $j = 1, \dots, k-1$.

2.3. Fixed point ratios and elements of prime order.

Definition 2.17. If a group G acts on a set Ω, then $C_\Omega(x)$ is the set of points in Ω fixed by $x \in G$. If G and Ω are finite, then the **fixed point ratio** of x, denoted by $\mathrm{fpr}(x)$, is the proportion of points in Ω fixed by x, i.e. $\mathrm{fpr}(x) = |C_\Omega(x)|/|\Omega|$.

If G acts transitively on a set Ω and H is a point stabiliser, then it is easy to see that

$$\mathrm{fpr}(x) = \frac{|x^G \cap H|}{|x^G|} \tag{2.9}$$

In [8, 9, 10, 11] Burness studies fixed point ratios in classical groups. Recall some observations from [8]. Let a group G act faithfully on the set Ω of right cosets of a subgroup H of G. Let $Q(G, c)$ be the probability that a randomly chosen c-tuple of points in Ω is not a base for G, so G admits a base of size c if and only if $Q(G, c) < 1$. Of course, a c-tuple is not a base if and only if it is fixed by $x \in G$ of prime order, and the probability that a random c-tuple is fixed by x is equal to $\mathrm{fpr}(x)^c$. Let $\mathscr{P}$ be the set of elements of prime order in G, and let $x_1, \dots, x_k$ be representatives for the G-classes of elements in $\mathscr{P}$. Since fixed point ratios are constant on conjugacy classes (see (2.9)),

$$Q(G, c) \leq \sum_{x \in \mathscr{P}} \mathrm{fpr}(x)^c = \sum_{i=1}^{k} |{x_i}^G| \cdot \mathrm{fpr}(x_i)^c =: \widehat{Q}(G, c). \tag{2.10}$$

Lemma 2.18 ([12, Lemma 2.1])**.** *Let G act faithfully and transitively on Ω and let H be a point stabiliser. If $x_1, \dots, x_k$ represent distinct G-classes such that $\sum_{i=1}^k |x_i^G \cap H| \leq A$ and $|x_i^G| \geq B$ for all $i \in \{1, \dots, k\}$, then*

$$\sum_{i=1}^{m} |x_i^G| \cdot \mathrm{fpr}(x_i)^c \leq B \cdot (A/B)^c.$$

for all $c \in \mathbb{N}$.

If there exists $\xi \in \mathbb{R}$ such that $\mathrm{fpr}(x) \leq |x^G|^{-\xi}$ for every $x \in \mathscr{P}$, then

$$\widehat{Q}(G,c) \leq \sum_{i=1}^{k} |x_i^G|^{1-c\xi}.$$

Definition 2.19. Let $\mathscr{C}$ be the set of conjugacy classes of prime order elements in G. For $t \in \mathbb{R}$,

$$\eta_G(t) := \sum_{C \in \mathscr{C}} |C|^{-t}.$$

If $Z(G) = 1$, then there exists $T_G \in \mathbb{R}$ such that $\eta_G(T_G) = 1$.

Lemma 2.20. *If G acts faithfully and transitively on Ω and $\mathrm{fpr}(x) \leq |x^G|^{-\xi}$ for all $x \in \mathscr{P}$ and $T_G < c\xi - 1$, then $b(G) \leq c$.*

Proof. We follow the proof of [12, Proposition 2.1]. Let $x_1, \ldots, x_k$ be representatives of the G-classes of prime order elements in G. By (2.10),

$$Q(G,c) \leq \sum_{i=1}^{k} |{x_i}^G| \cdot \mathrm{fpr}(x_i)^c \leq \eta_G(c\xi - 1).$$

The result follows since $\eta_G(t) < 1$ for all $t > T_G$. □

We fix the following notation for the rest of the section. Let $\overline{G}$ be an adjoint simple algebraic group of type A_{n-1} or $C_{n/2}$ over the algebraic closure of $\mathbb{F}_p$. Let $\overline{G}_\sigma = \{g \in \overline{G} \mid g^\sigma = g\}$ where σ is a Frobenius morphism of $\overline{G}$. Let $\overline{G}$ be such that $G_0 = O^{p'}(\overline{G}_\sigma)'$ is a finite simple group. Here $O^{p'}(G)$ is the subgroup of a finite group G generated by all p-elements of G. Therefore, $\overline{G}_\sigma = \mathrm{PGL}_n^\varepsilon(q)$ and $G_0 = \mathrm{PSL}_n^\varepsilon(q)$ for type A_{n-1}; also $\overline{G}_\sigma = \mathrm{PGSp}_n(q)$ and $G_0 = \mathrm{PSp}_n(q)'$ for type $C_{n/2}$. Let G be a finite almost simple group with socle G_0.

As proved in [12, Proposition 2.2], if $n \geq 6$, then T_G exists and $T_G < 1/3$. Thus, if for such G

$$\mathrm{fpr}(x) < |x^G|^{-\frac{4}{3c}} \tag{2.11}$$

for all $x \in \mathscr{P}$, then $\xi \geq 4/(3c)$ and $c\xi - 1 \geq 1/3 > T_G$ and G has a base of size c.

Therefore, Lemma 2.20 allows us to estimate the base size by calculating bounds for $|x^G|$ and $|x^G \cap H|$ for elements x of prime order.

Definition 2.21. Let $x \in \mathrm{PGL}(V) = \mathrm{PGL}_n(q)$. Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}_q$, and let $\overline{V} = \overline{\mathbb{F}} \otimes V$. Let $\hat{x}$ be the preimage of x in $\mathrm{GL}_n(q)$. Define

$$\nu_{V,\overline{\mathbb{F}}}(x) := \min\{\dim[\overline{V}, \lambda\hat{x}] : \lambda \in \overline{\mathbb{F}}^*\}.$$

Here $[V,g]$ for a vector space V and $g \in \mathrm{GL}(V)$ is the commutator in $V \rtimes \mathrm{GL}(V)$. Therefore, $\nu_{V,\overline{\mathbb{F}}}(x)$ is the minimal codimension of an eigenspace of $\hat{x}$ on $\overline{V}$. Sometimes we denote this number by $\nu(x)$ and $\nu_{V,\overline{\mathbb{F}}}(\hat{x})$.

Lemma 2.22 ([9, Lemma 3.11])**.** *Let $x \in \mathrm{PGL}_n^\varepsilon(q)$ have prime order r. One of the following holds:*

(1) *x lifts to $\hat{x} \in \mathrm{GL}_n^\varepsilon(q)$ of order r such that $|x^{\mathrm{PGL}_n^\varepsilon(q)}| = |\hat{x}^{\mathrm{GL}_n^\varepsilon(q)}|$;*

(2) *r divides both $q - \varepsilon$ and n, and x is $\mathrm{PGL}_n(\overline{\mathbb{F}})$-conjugate to the image of*

$$\mathrm{diag}[I_{n/r}, \omega I_{n/r}, \ldots, \omega^{r-1} I_{n/r}],$$

where $\omega \in \overline{\mathbb{F}}$ is a primitive r-th root of unity.

Remark 2.23. Lemma 3.11 from [9] is formulated for all classical groups, but only for $r \neq 2$. It is easy to see from its proof that the condition $r \neq 2$ is necessary only for orthogonal and symplectic cases; if $x \in \mathrm{PGL}_n^{\varepsilon}(q)$ then the statement is true for arbitrary prime $|x|$.

Lemma 2.24. *Let $x \in \overline{G}_\sigma$ have prime order.*

1. *If x is semisimple, then $x^{\overline{G}_\sigma} = x^{G_0}$.*
2. *If x is unipotent and $G_0 = \mathrm{PSL}_n^{\varepsilon}(q)$, then $|x^{\overline{G}_\sigma}| \leq \min\{n, p\}|x^{G_0}|$.*
3. *If x is unipotent, $p \neq 2$ and $G_0 = \mathrm{PSp}_n(q)$, then $|x^{\overline{G}_\sigma}| \leq 2|x^{G_0}|$.*

Proof. See [25, 4.2.2(j)] for the proof of (1) and [9, Lemma 3.20] for (2) and (3). □

Notice that if $p = 2$, $(n, q) \neq (4, 2)$ and $\overline{G}_\sigma$ is symplectic, then $\overline{G}_\sigma = G_0$.

Lemma 2.25. *Let $x \in G$ have prime order r and $s := \nu(x)$.*

1. *In case $G_0 = \mathrm{PSL}_n^{\varepsilon}(q)$:*
$$|x^G| > \begin{cases} \frac{1}{2t}\left(\frac{q}{q+1}\right)^{as/(n-s)} q^{ns} \geq \frac{1}{2t}\left(\frac{q}{q+1}\right)^{as/(n-s)} q^{n^2/2} & \text{for } s \geq n/2; \\ \frac{1}{2t}\left(\frac{q}{q+1}\right)^{a} q^{2s(n-s)} \geq \frac{1}{2t}\left(\frac{q}{q+1}\right)^{a} q^{(3/8)n^2} & \text{for } n/4 \leq s < n/2, \end{cases} \tag{2.12}$$
where $t = \min\{r, n\}$ and $a = (1/2)(1 - \varepsilon 1)$.
2. *In case $G_0 = \mathrm{PSp}_n(q)'$:*
$$|x^G| > \frac{1}{8}\left(\frac{q}{q+1}\right)\max(q^{s(n-s)}, q^{(ns/2)}). \tag{2.13}$$

Proof. The statement follows by Lemma 2.24 and [9, Propositions 3.22 and 3.36, Lemmas 3.34 and 3.38]. □

3. Intersection of conjugate irreducible solvable subgroups

Recall from the introduction that $b_S(G)$ is the minimal number such that there exist
$$x_1, \dots, x_{b_S(G)} \in G \text{ with } S^{x_1} \cap \dots \cap S^{x_{b_S(G)}} = S_G$$
where $S_G = \cap_{g \in G} S^g$. Let G be $\mathrm{GU}_n(q)$ or $\mathrm{GSp}_n(q)$ in cases **U** and **S** respectively and let S be an irreducible maximal solvable subgroup of G. The goal of this section is to obtain upper bounds for $b_S(S \cdot (\mathrm{SL}_n(q^{\mathbf{u}}) \cap G))$. These bounds play an important role in the proof of Theorems B1 and C1 in Section 4. While, with some exceptions, $b_S(S \cdot (\mathrm{SL}_n(q^{\mathbf{u}}) \cap G)) \leq 4$ follows by [12, Theorem 1.1], it is not sufficient for our purposes. In this section we prove that $b_S(S \cdot (\mathrm{SL}_n(q^{\mathbf{u}}) \cap G)) \leq 3$ in cases **U** and **S** with a short list of exceptions.

Maximal solvable subgroups of $\mathrm{GSp}_n(q)$ were recently classified in [31]. For our purpose, we do not need the classification in full details, but we make use of it when n is small. In particular, [31, Table 14] lists, up to conjugacy, irreducible maximal solvable subgroups of $\mathrm{GSp}_n(q)$ for $n \in \{2, 4, 6\}$.

3.1. Primitive and quasi-primitive subgroups.

We start our study with a special case when S is quasi-primitive solvable subgroup of G. In the next section we use these results to obtain bounds for $b_S(S \cdot (\mathrm{SL}_n(q^{\mathbf{u}}) \cap G))$ where S is irreducible.

Definition 3.1. Let $H \leq \mathrm{GL}(V)$. An irreducible $\mathbb{F}_q[H]$-module V is **quasi-primitive** if it is a homogeneous $\mathbb{F}_q[N]$-module for all $N \trianglelefteq H$. A subgroup H of $\mathrm{GL}(V)$ is **quasi-primitive** if V is a quasi-primitive $\mathbb{F}_q[H]$-module.

In the linear case studied in [5], to extend our results from a primitive subgroup to an irreducible subgroup, we use [37, §18, Theorem 5] which asserts that an imprimitive irreducible solvable subgroup of $\mathrm{GL}_n(q)$ lies in a wreath product of a primitive solvable subgroup of smaller degree with a solvable permutation group. In cases **U** and **S**, Lemma 2.6 does not guarantee that an irreducible subgroup of G lies in the wreath product of a primitive subgroup of a general unitary or symplectic group of smaller degree with a subgroup of a symmetric group. However, it gives us a decomposition of V which allows us to use induction when S is not quasi-primitive.

To prove results about $b_S(\mathrm{SL}_n(q^{\mathbf{u}}) \cap G)$, we need information about quasi-primitive solvable groups such as upper bounds for $|S|$ and lower bound for $\nu(x)$ (the codimension of a largest eigenspace of x, see Definition 2.21), where x is a prime order element of the image of S in $\mathrm{PGL}_n(q^{\mathbf{u}})$. This information is needed to apply the probabilistic method described in Section 2.3. A primitive subgroup of $\mathrm{GL}_n(q)$ is quasi-primitive by Clifford's Theorem. If $S \leq \mathrm{GL}_n(q)$ is solvable and quasi-primitive, then every normal abelian subgroup of S is cyclic by [33, Lemma 0.5]. Such groups are studied in [33]; we collect the main results in the following lemma.

Lemma 3.2 ([33, Corollary 1.10])**.** *Suppose $S \leq \mathrm{GL}_n(q)$ is nontrivial solvable and every normal abelian subgroup of S is cyclic. Let $F = \mathbf{F}(S)$ be the Fitting subgroup of S and let Z be the socle of the cyclic group $Z(F)$. Set $C = C_S(Z)$. Then there exist normal subgroups E and T of S satisfying the following:*

(1) *$F = ET$, $Z = E \cap T$ and $T = C_F(E)$;*
(2) *$E/Z = E_1/Z \times \ldots \times E_k/Z$ for chief factors E_i/Z of G with $E_i \leq C_S(E_j)$ for $i \neq j$;*
(3) *For each i, $Z(E_i) = Z$, $|E_i/Z| = p_i^{2k_i}$ for a prime p_i and an integer k_i, and $E_i = O_{p_i'}(Z) \cdot F_i$ for an extra-special group $F_i = O_{p_i}(E_i) \trianglelefteq S$ of order $p_i^{2k_i+1}$;*
(4) *There exists $U \leq T$ of index at most 2 with U cyclic, $U \trianglelefteq S$ and $C_T(U) = U$;*
(5) *$T = C_S(E)$ and $F = C_C(E/Z)$;*
(6) *If C_i is the centraliser of E_i/Z in C, then C/C_i is isomorphic to a subgroup of $\mathrm{Sp}_{2k_i}(p_i)$.*

Remark 3.3. In the notation of Lemma 3.2, let e be a positive integer such that $|E/Z| = e^2$, so $e = \prod_{i=1}^{k} p_i^{k_i}$. Since E_i has the subgroup F_i of order $p_i^{2k_i+1}$ and $|E_i/Z| = p_i^{2k_i}$, for each p_i there must exist an element of order p_i in Z (and in U, since $Z \leq U$). In other words, each p_i divides $|U|$.

Theorem 3.4 ([33, Theorem 3.5])**.** *If S is a completely reducible solvable subgroup of $\mathrm{GL}_n(q)$, then $|S| < q^{9n/4}/2.8$.*

First we obtain results about cyclic subgroups of $G \in \{\mathrm{GU}_n(q), \mathrm{GSp}_n(q)\}$. To do so, in particular, we use the notion of a **Singer cycle**: a cyclic subgroup of $\mathrm{GL}_n(q)$ of order $q^n - 1$. We use notation and results about Singer cycles from [5, Section 2]. The following result is well known and is easy to prove using Clifford's Theorem and linear algebra.

Lemma 3.5. *An irreducible cyclic subgroup of $\mathrm{GL}_n(q)$ is contained in some Singer cycle. A proper subgroup C of a Singer cycle $T \leq \mathrm{GL}_n(q)$ is irreducible if and only if $|C|$ does not divide $q^r - 1$ for every proper divisor r of n.*

Lemma 3.6. *Let C be a non-scalar cyclic subgroup of $G \in \{\mathrm{Sp}_n(q), \mathrm{GU}_n(q)\}$ such that V is $\mathbb{F}_{q^{\mathbf{u}}}[C]$-homogeneous. Recall that $\mathbf{u} = 2$ if $G = \mathrm{GU}_n(q)$ and $\mathbf{u} = 1$ otherwise. If $W \subseteq V$ is a $\mathbb{F}_{q^{\mathbf{u}}}[C]$-irreducible C-invariant subspace with $\dim W = m$, then $|C|$ divides $(q^{\mathbf{u}})^{m/2}+1$. Moreover, m is even if $G = \mathrm{Sp}_n(q)$ and m is odd if $G = \mathrm{GU}_n(q)$.*

Proof. Since W is $\mathbb{F}_{q^{\mathbf{u}}}[C]$-irreducible, it is either non-degenerate or totally isotropic. If W is non-degenerate, then the lemma follows by [28, Satz 4 and 5].

Let W be totally isotropic. We consider here the proof for $G = \mathrm{Sp}_n(q)$, the proof for $\mathrm{GU}_n(q)$ is analogous. By [1, (5.2)], we can assume that there exist $W_1, W_2 \subseteq V$ such that $W_1 = W$ and W_2 is totally isotropic and $\mathbb{F}_{q^{\mathbf{u}}}[C]$-irreducible, and $W_1 \oplus W_2$ is non-degenerate.

Let $g \in Sp(W_1 \oplus W_2)$ be the restriction of a generator of C to $W_1 \oplus W_2$. Since V is $\mathbb{F}_{q^{\mathbf{u}}}[C]$-homogeneous, there exist bases β_i of W_i for $i = 1, 2$ (let $\beta = \beta_1 \cup \beta_2$) such that

$$g_\beta = \begin{pmatrix} g_1 & 0 \\ 0 & g_1 \end{pmatrix} \text{ and } (\mathbf{f}|_{W_1 \oplus W_2})_\beta = \begin{pmatrix} 0 & A \\ -A^\top & 0 \end{pmatrix}$$

for some $g_1, A \in \mathrm{GL}_m(q)$. Here $\langle g_1 \rangle \leq \mathrm{GL}_m(q)$ is irreducible. Since g is an isometry of $W_1 \oplus W_2$,

$$g_\beta (\mathbf{f}|_{W_1 \oplus W_2})_\beta (g_\beta)^\top = (\mathbf{f}|_{W_1 \oplus W_2})_\beta,$$

so $g_1 A g_1^\top = A$ and $g_1^A = (g_1^{-1})^\top$. Therefore, the set of eigenvalues of g_1 is closed under taking inverses. Moreover, the multiplicity of the eigenvalue μ is equal to the multiplicity of the eigenvalue μ^{-1} for every $\mu \in \overline{\mathbb{F}_q}^*$.

By [18, Lemma 1.3], g_1 is conjugate in $\mathrm{GL}_m(\overline{\mathbb{F}_q})$ to

$$\mathrm{diag}(\lambda, \lambda^q, \ldots, \lambda^{q^{m-1}}), \text{ where } \lambda^{q^m - 1} = 1.$$

Clearly, $|\lambda| = |g_1| = |C|$. Let r be the minimal natural number such that $\lambda^{q^r} = \lambda^{-1}$. If $r = 0$, so $\lambda = \lambda^{-1} = \pm 1$, then $g_1 = \lambda I_m$ and C is a group of scalars, since it is homogeneous. Assume $r > 0$, so $\lambda^{q^r + 1} = 1$ and $(q^r + 1)$ divides $(q^m - 1)$. Since $\langle g_1 \rangle$ is irreducible, $|g_1|$ does not divide $q^l - 1$ for every proper divisor l of m. Notice that $|g_1|$ divides $(q^{2r} - 1)$, so $|g_1|$ divides $(q^{2r} - 1, q^m - 1) = q^{(2r, m)} - 1$ which is divisible by $q^r + 1$. Therefore, $(2r, m) > r$ and $2r = m$. Hence m is even and $|C|$ divides $q^{m/2} + 1$. □

Corollary 3.7. *Let C be a non-scalar cyclic subgroup of $\mathrm{GSp}_n(q)$ such that V is $\mathbb{F}_q[C]$-homogeneous. If $W \subseteq V$ is a $\mathbb{F}_q[C]$-irreducible submodule of dimension m, then m is even and $|C|$ divides $(q^{m/2} + 1)(q - 1)$.*

Proof. Let $C = \langle c \rangle$ and $\tau(c) = \lambda \in \mathbb{F}_q$ where τ is as in Definition 2.3. So

$$(uc, vc) = \lambda(u, v) \text{ for all } u, v \in V.$$

Notice that

$$(uc^{|\lambda|}, vc^{|\lambda|}) = \lambda^{|\lambda|}(u, v) = (u, v) \text{ for all } u, v \in V.$$

Therefore, $c^{|\lambda|} \in \mathrm{Sp}_n(q)$. Let c_1 be the restriction of c to W, so $c = \mathrm{diag}[c_1, \ldots, c_1]$ in some basis of V since V is $\mathbb{F}_q[C]$-homogeneous.

We claim that $\langle c_1^{|\lambda|} \rangle$ is an irreducible subgroup of $\mathrm{GL}(W)$. Assume the opposite, so there exists a $\langle c_1^{|\lambda|} \rangle$-invariant subspace of W of dimension r dividing m. Hence r is even and $|c_1|/|\lambda|$ divides $(q^{r/2} + 1)$ by Lemma 3.6 since $c^{|\lambda|} \in \mathrm{Sp}_n(q)$. Also $|\lambda|$ divides $(q - 1)$ and $(q - 1)$ divides $(q^{r/2} - 1)$, so $|c_1| = |\lambda| \cdot |c^{|\lambda|}|$ divides $q^r - 1$. By Lemma 3.5, $\langle c_1 \rangle$ is a reducible subgroup of $\mathrm{GL}(W)$ which is a contradiction.

Thus, W is $\langle c_1^{|\lambda|} \rangle$-irreducible and $|c^{|\lambda|}|$ divides $(q^{m/2} + 1)$ by Lemma 3.6, so $|C|$ divides $(q^{m/2} + 1)(q - 1)$. □

Now we obtain bounds for $|S|$ and $\nu(x)$. We adopt the notation of Lemma 3.2 in the following statement.

Lemma 3.8. *Let $G \in \{\mathrm{GSp}_n(q), \mathrm{GU}_n(q)\}$. Let S be a quasi-primitive solvable subgroup of G. Let W be an m-dimensional irreducible U-submodule of V and let e be a positive integer such that $e^2 = |E/Z|$. The following hold:*

(1) *em divides n;*
(2) $|S| \leq \min\{|U|^2 e^{13/2}/2, |U|me^{13/2}\}$;
(3) *if $e = 1$, then $n = m$ and S is a subgroup of the normaliser of a Singer cycle of* $\mathrm{GL}_n(q^{\mathbf{u}})$;
(4) *if $m = 1$ then* $|S| \leq |Z(G)|e^{13/2}$.

Proof. Since $EU \trianglelefteq S$, (1) follows by Clifford's Theorem and [33, Corollary 2.6].

It is easy to see that

$$|S| = |S/C| \cdot |T| \cdot |C/F| \cdot |F/T|.$$

By the proof of [33, Corollary 3.7], $|S/C| \cdot |T| \leq |U|^2$, and $|C/F| \leq e^{9/2}/2$ and $|F/T| = e^2$, which gives us the first bound of (2). To obtain the second bound, we claim that $|S/C| \leq m$. Indeed, the linear span $\mathbb{F}_{q^{\mathbf{u}}}[Z]$ is the field extension K of the field of scalar matrices $\Delta = \mathbb{F}_{q^{\mathbf{u}}} \cdot I_n$ of degree $m_1 = \dim W_1$, where $W_1 \leq V$ is an irreducible $\mathbb{F}_{q^{\mathbf{u}}}[Z]$-module, so m_1 divides n since Z is homogeneous, and $m_1 \leq m$ since $Z \leq U$. Consider the map

$$f : S \to \mathrm{Gal}(K/\Delta),\ g \mapsto \sigma_g,$$

where $\sigma_g : K \to K$, $x^{\sigma_g} = x^g$ for $x \in K$. Since $\ker(f) = C$,

$$S/C \cong \mathrm{Im}(f) \leq \mathrm{Gal}(K/\Delta),$$

so $|S/C|$ divides m_1 and the second bound follows.

If $e = 1$, then $F = T$ and S is a subgroup of the normaliser of a Singer cycle of $\mathrm{GL}_n(q^{\mathbf{u}})$ by [33, Corollary 2.3], so U is self-centralising. By [33, Lemma 2.2] U is irreducible, so $m = n$ and (3) follows.

If $m = 1$, then $U \leq Z(G)$ since U is homogeneous, so $|S/C| = 1$ which implies (4). □

Lemma 3.9. *Let S be a quasi-primitive solvable subgroup of $\mathrm{GL}_n(q)$ and let H be the image of S under the natural homomorphism from $\mathrm{GL}_n(q)$ to $\mathrm{PGL}_n(q)$. If $x \in H$ has prime order, then $\nu(x) \geq n/4$.*

Proof. The proof follows the beginning of the proof of [23, Proposition 4]. Let $\hat{x}$ be a preimage of x in $\mathrm{GL}_n(q)$, so $\hat{x} = \mu g$, where $\mu \in Z(\mathrm{GL}_n(q))$ and $g \in S\backslash\{1\}$. Observe that $\nu(x) = \nu(\hat{x}) = \nu(g)$, so it suffices to prove that $\nu(g) \geq n/4$ for all nontrivial $g \in S$.

If $g \in U$, then, since U is abelian and V is U-homogeneous, g is conjugate in $\mathrm{GL}_n(\overline{\mathbb{F}_q})$ to

$$\mathrm{diag}(\lambda, \lambda^q, \ldots, \lambda^{q^{m_1-1}}, \ldots, \lambda, \lambda^q, \ldots, \lambda^{q^{m_1-1}});\ \lambda \in \overline{\mathbb{F}_q}$$

by [18, Lemma 1.3]. Here m_1 is the smallest possible integer such that $\lambda^{q^{m_1}} = \lambda$. Therefore, $\nu(g) = n - n/m_1 \geq n/2$. Moreover, if $z \in U$ is nontrivial, then $C_V(z) = \{0\}$.

Let $\lambda \in \overline{\mathbb{F}_q}^*$. If $g \in S\backslash C$, then

$$[\lambda g, z] = [g, z] \in Z\backslash\{1\}$$

for some $z \in Z \leq U$. Notice

$$C_V((\lambda g)^{-1}) \cap C_V(z^{-1}\lambda g z) \subseteq C_V([\lambda g, z]) = \{0\}$$

and $\dim(C_V((\lambda g)^{-1})) = \dim(C_V((\lambda g)^z))$, so $\dim(C_V((\lambda g))) \leq n/2$ for every $\lambda \in \overline{\mathbb{F}_q}^*$. Hence $\nu(g) \geq n/2$.

If $g \in F\backslash T$, then, by (1) and (5) of Lemma 3.2, there exists $h \in E$ such that

$$[\lambda g, h] = [g, h] \in Z\backslash\{1\}$$

and $\nu(g) \geq n/2$ as above.

If $g \in T\backslash U$, then $[\lambda g, u] = [g, u] \in U\backslash\{1\}$ for some $u \in U$ by (4) of Lemma 3.2, so $\nu(g) \geq n/2$ as above.

If $g \in C\backslash F$, then $[\lambda g, h] \in E\backslash Z \subseteq F\backslash T$ for some $h \in E$ by (5) of Lemma 3.2. Therefore,

$$\dim(C_V([\lambda g, h])) \leq n/2 \text{ and } \dim(C_V(\lambda g)) \leq 3n/4$$

for every $\lambda \in \overline{\mathbb{F}_q}^*$, so $\nu(g) \geq n/4$. □

Before the next lemma, we remind the reader the following definitions. A subgroup of $\mathrm{GL}_n(q)$ is **absolutely irreducible** if it is irreducible as a subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}_q})$, where $\overline{\mathbb{F}_q}$ is the algebraic closure of $\mathbb{F}_q$. For a prime r, a finite r-group R is of **symplectic-type** if every characteristic abelian subgroup of R is cyclic. Symplectic-type groups are closely related to extra-special groups, see [30, §4.6] and [36, §2.4] for a summary and details. Normalisers of absolutely irreducible symplectic-type subgroups in classical groups are often maximal subgroups. Such maximal subgroups form Aschbacher's class $\mathcal{C}_6$. If such a normaliser is not a maximal subgroup, then it lies in a maximal subgroup from the class $\mathcal{C}_5$ (which contains the normalisers of classical subgroups over a subfield of $\mathbb{F}_{q^{\mathbf{u}}}$, see [30, §4.5, 4.6] for details). We use the notation of Lemma 3.2 in the following lemma.

Lemma 3.10. *Let $S \leq \hat{G} \in \{\mathrm{GU}_n(q), \mathrm{GSp}_n(q)\}$ be a quasi-primitive maximal solvable subgroup. Recall that $q = p^f$. If $e = n = r^l$ for some integer l and prime r, then the following hold:*

1. $T = Z(F) = C_S(E) = Z(\hat{G})$;
2. $S = S_1 \cdot Z(\hat{G})$ *where* $S_1 = S \cap \mathrm{GL}_n(p^t)$, *$t$ divides f, and S_1 lies in the normaliser M in* $\mathrm{GL}_n(p^t)$ *of an absolutely irreducible symplectic-type subgroup of* $\mathrm{GL}_n(p^t)$.

Proof. By [33, Lemma 2.10], (1) follows. Let $W \leq V$ be an irreducible $\mathbb{F}_{q^{\mathbf{u}}}[F]$-submodule. By Theorem 3.2, $F = O_{r'}(Z) \cdot F_1$ where F_1 is extra-special of order r^{2l+1}, so W is a faithful irreducible $\mathbb{F}_{q^{\mathbf{u}}}[F_1]$-module. Therefore, by [30, Proposition 4.6.3], $\dim W = r^l$, and F_1 is an absolutely irreducible subgroup of $\hat{G} \cap \mathrm{GL}_n(p^t)$ where $\mathbb{F}_{p^t}$ is the smallest field over which such a representation of F_1 can be realised. In particular, t is the smallest positive integer for which $p^t \equiv 1 \mod |Z(F_1)|$. Moreover, by [36, Corollary 2.4.12], $|Z(F_1)|$ is either r or 4. By [36, Theorem 2.4.12], $S = S_1 \cdot Z(\hat{G})$ where $S_1 \leq N_{\mathrm{GL}_n(p^t)}(F_1) = M$. □

Finally, we establish the bounds for b_S.

Lemma 3.11.

1. *Let $n \geq 3$ and $(n, q) \neq (3, 2)$. If S is a quasi-primitive maximal solvable subgroup of* $\mathrm{GU}_n(q)$, *then*
$$b_S(S \cdot \mathrm{SU}_n(q)) \leq 3.$$
2. *Let $n \geq 6$. If S is a quasi-primitive maximal solvable subgroup of* $\mathrm{GSp}_n(q)$, *then*
$$b_S(S \cdot \mathrm{Sp}_n(q)) \leq 3.$$

Proof. Let $\hat{G}$ be $S \cdot \mathrm{SU}_n(q)$ and $S \cdot \mathrm{Sp}_n(q)$, for cases (1) and (2) respectively. Let $G = \hat{G}/Z(\hat{G}) \leq \mathrm{PGL}_n(q^{\mathbf{u}})$ and let H be $S/Z(\hat{G}) \leq G$. Obviously,

$$b_S(\hat{G}) = b_H(G).$$

If $n \geq 6$ and for all $x \in G$ of prime order

$$|x^G \cap H| < |x^G|^{(3c-4)/(3c)},$$

then $b_H(G) \le c$ by Lemma 2.9 and (2.11). Therefore, if $n \ge 6$, then it suffices to show this inequality with $c = 3$.

Let $s := \nu(x)$. We use bounds (2.12) and (2.13) for $|x^G|$. In most cases the bound $|x^G \cap H| \le |H|$ is sufficient.

Part (1) of the lemma follows from (2.11), Lemma 3.9, bounds (2.12), Lemma 3.6 and (2) of Lemma 3.8 for all q and for $n \ge 10$. These bounds do not suffice when $6 \le n \le 9$ and $q = 2$; here the lemma is verified by computation. For $n \le 5$, with a finite number of exceptions verified by computation, part (2) follows by [12, Tables 2 and 3] where bounds for base sizes for primitive actions of classical groups with $n \le 5$ are listed. For $n \ne 4$, an upper bound for the base size is listed for all (up to conjugacy) irreducible maximal subgroups H. If $n = 4$, and the maximal subgroup H is of type $\mathrm{Sp}_4(q)$, then the corresponding action is equivalent to a subspace action of an orthogonal group with socle $\mathrm{P}\Omega_6^-(q)$. In this case, in the notation of Lemma 3.8, $m = 1$ by Lemma 3.6 and $e = 4$ by [33, Corollary 2.6], so S is as in Lemma 3.10. So S lies in the maximal subgroup of H of type $2^4.\mathrm{O}_4^-(2)$ and $b_S(S \cdot \mathrm{SU}_4(q)) \le 3$ by [12, Table 3].

Part (2) of the lemma follows from (2.11), Lemma 3.9, bounds (2.12), Corollary 3.7 and (2) of Lemma 3.8 for all q and for $n \ge 16$. The list of cases when these bounds are not sufficient for $6 \le n \le 14$ is finite. Using Remark 3.3 we reduce this list to $6 \le n \le 8$ and $q \in \{2, 3, 5, 7\}$; here the lemma is verified by computation. □

Theorem 3.12. *Let G be $\mathrm{GU}_n(q)$ or $\mathrm{GSp}_n(q)$ in cases* **U** *and* **S** *respectively. Let S be a quasi-primitive maximal solvable subgroup of G. In each case let (n, q) be such that G is not solvable. Then either $b_S(S \cdot (\mathrm{SL}_n(q^{\mathbf{u}}) \cap G)) \le 3$, or $G \in \{\mathrm{GU}_2(5), \mathrm{GSp}_2(5)\}$, S is a completely irreducible subgroup with $S/Z(G)$ isomorphic to $2^2.\mathrm{Sp}_2(2)$ and $b_S(S \cdot \mathrm{SL}_2(5)) = 4$.*

Proof. Lemma 3.11 gives us sufficient results for $n \ge 3$ in case **U** and for $n \ge 6$ in case **S**. If $n = 2$, then $\mathrm{GSp}_n(q) = \mathrm{GL}_2(q)$ and $\mathrm{PGU}_n(q) \cong \mathrm{PGL}_2(q)$, so, by Lemma 2.13, $b_S(S \cdot (\mathrm{SL}_n(q^{\mathbf{u}}) \cap G)) \le 3$ unless $q = 5$. For $q = 5$, in both cases **U** and **S**, the lemma is verified by computations.

Let S be a quasi-primitive maximal solvable subgroup of $\mathrm{GSp}_4(q)$ and $\hat{G} = S \cdot \mathrm{Sp}_4(q)$. We use notation from Lemmas 3.2 and 3.8. The proof splits into several cases depending on values of e, m, and q.

<u>*Case $e = 4$.*</u> If $e = 4$, then $m = 1$, q is odd by Remark 3.3 and S is as in Lemma 3.10. In particular, S lies in a maximal subgroup of $\hat{G}$ from $\mathcal{C}_5$ or $\mathcal{C}_6$. By [12, Tables 2 and 3], $b_S(S \cdot \mathrm{Sp}_4(q)) \le 3$ for $q > 3$; for $q = 3$ the statement $b_S(S \cdot \mathrm{Sp}_4(q)) \le 3$ is established by computation.

Let G and H be the image of $\hat{G}$ and S in $\mathrm{PGSp}_4(q)$ under the natural homomorphism respectively. In the remaining cases we claim that $Q(G, 3) < 1$ in (2.10).

<u>*Case $e = 1$.*</u> In this case, $m = 4$ and S lies in the normaliser $N = T \rtimes \langle \varphi \rangle$ of a Singer cycle T of $\mathrm{GL}_4(q)$ by Lemma 3.8(3). Here $|\varphi| = 4$ and $t^\varphi = t^q$ for $t \in T$ by [27, Chapter II, §7]. In particular, $S \cap T$ is irreducible and $|S \cap T|$ divides $(q^2 + 1)(q - 1)$ by Corollary 3.7. So $|H| \le 4(q^2 + 1)$. By the proof of Lemma 3.9, $\nu(x) \ge n/2 = 2$ for all elements $x \in H$ of prime order.

All elements of of H having odd prime order lie in the image of $S \cap T$ in H, so there are at most $A_1 := (q^2 + 1)$ such elements. By [9, Lemma 3.34 and Proposition 3.36],

$$|x^G| > (1/2)q^4 =: B_1$$

for semisimple $x \in H$ of prime order.

If q is even, then, since $N = T \rtimes \langle\varphi\rangle$ and $\langle\varphi\rangle$ is a Sylow 2-subgroup of N, all elements of H of order 2 are conjugate in H, so there are at most q^2+1 such elements. Assume that q is odd. Let us compute the number of elements $(\lambda\varphi^i) \in S \leq N$ such that $(\lambda\varphi^i)^2$ is scalar, so $(\lambda\varphi^i)^2 \in Z(\hat{G}) = Z(\mathrm{GSp}_n(q))$. Notice that $|Z(\mathrm{GSp}_n(q))| = |Z(\mathrm{GL}_n(q))| = q-1$. Since

$$(\lambda\varphi^i)^2 = \lambda^{q^{4-i}+1}\varphi^{2i} \in Z(\hat{G}),$$

there are two possibilities: $i = 4$ and $i = 2$. If $i = 4$, then $\lambda^2 \in Z(\hat{G})$, so there are $2(q-1)$ such elements in T. In the second case $\lambda^{q^2+1} \in Z(\hat{G})$, so there are $(q^2+1)(q-1)$ such elements. Therefore, there are at most $((q^2+1)+2)-1 = q^2+2 =: A_2$ elements of order two in H. By Lemma 2.25, $|x^G| > (1/8)q^5/(q+1) := B_2$.

Hence, by Lemma 2.18, $Q(G,3) \leq A_1^3/B_1^2 + A_2^3/B_2^2$ which is less than 1 for $q > 9$, so $b_H(G) \geq 3$. If $q < 9$, then $b_S(\hat{G}) \leq 3$ is established by computation.

<u>*Case $e = 2$.*</u> Note that q is odd by Remark 3.3. First, let us show that we can assume that S is primitive. Indeed, if S is imprimitive, then there exists a system of imprimitivity

$$V = V_1 \oplus \ldots \oplus V_k.$$

Let k be the maximum possible for S, so $k \in \{2,4\}$. By carefully examining [31, Table 14] (where, as mentioned in the beginning of the section, listed irreducible maximal solvable subgroups of $\mathrm{GSp}_n(q)$ for small n) and using [31, Theorem 24.9], we obtain that $k = 2$ and the V_i are totally isotropic, as otherwise S it is metrically imprimitive and not quasi-primitive. Hence S lies in a maximal group of $\mathrm{GSp}_4(q)$ of type $\mathrm{GL}_2(q).2$ (see [30, Table 4.2.A]) and $b_S(\hat{G}) \leq 3$ by [12, Table 2]. So further we assume that S is primitive.

Since

$$|S| = |S/C| \cdot |T/U| \cdot |U| \cdot |C/F| \cdot |F/T|,$$

by (6) of Lemma 3.2, $|S|$ divides $2 \cdot 2 \cdot (q^2-1) \cdot |\mathrm{Sp}_2(2)| \cdot e^2$. Therefore, $|H|$ divides $96(q+1)$. Let $x \in H$ have prime order r. Let Q_1, Q_2 and Q_3 be

$$\sum_{x\in\mathscr{P};r\notin\{2,3\}} \mathrm{fpr}(x)^3, \quad \sum_{x\in\mathscr{P};r=2} \mathrm{fpr}(x)^3 \text{ and } \sum_{x\in\mathscr{P};r=3} \mathrm{fpr}(x)^3$$

respectively, so $Q(G,3) \leq Q_1 + Q_2 + Q_3$. We find upper bounds for Q_i, $i \in \{1,2,3\}$.

If $r \neq 2,3$ then r divides $q+1$ and, since r does not divide n, by Lemma 2.22 x has a preimage $\hat{x} \in U$ of order r. Hence, by the proof of Lemma 3.9, $\nu(x) \geq 2$. Since U is a normal cyclic subgroup of order dividing $q+1$, the number of elements of H of prime power not equal to 2 or 3 is at most $A_1 := q$. By [9, Proposition 3.36], $|x^G| > (1/2)q^4 =: B_1$. By Lemma 2.18

$$Q_1 \leq A_1^2/B_1^2 = 1/(4q^5).$$

We use results summarised in the following remark to estimate $|x^G \cap H|$ for $r \in \{2,3\}$.

Remark 3.13. Let $r \in \{2,3\}$. For $L \leq \mathrm{GL}_n(q)$ let

$$c_r(L) = |\{g \in L : g^r \in Z(\mathrm{GL}_n(q))\}|.$$

Notice that $|x^G \cap H| \leq c_r(S)/|Z(\mathrm{Sp}_4(q))|$ for $x \in S/Z(\mathrm{Sp}_4(q))$ of prime order. Here we compute $c_r(L)$ for some specific groups.

By [37, §21, Theorem 6] and [36, Chapter 5], a primitive maximal solvable subgroup of $\mathrm{GL}_2(q)$ is conjugate to either the normaliser of a Singer cycle or to a certain subgroup of order $24(q-1)$. We follow [36, Chapter 5] and denote the normaliser of a Singer cycle of $\mathrm{GL}_2(q)$ by M_2 and the primitive maximal solvable subgroup of order $24(q-1)$ by M_3 and M_4 for $q \equiv 3 \bmod 4$ and

$q \equiv 1 \bmod 4$ respectively. Explicit generating sets of M_3 and M_4 are listed in [36, §5.2]. It is routine to check that M_3 and M_4 contain $Z(\mathrm{GL}_2(q))$, $c_2(M_i) = 10(q-1)$ and $c_3(M_i) = 9(q-1)$ for both $i = 3, 4$. Notice that $c_3(M_2) = (3, q+1)\cdot(q-1)$, since all $g \in M_2$ such that $g^3 \in Z(\mathrm{GL}_2(q))$ lie in the Singer cycle which is a normal subgroup of M_2. Using the same method as for the case $m = 4$ (when S lies in the normaliser of a Singer cycle), we obtain $c_2(M_2) = (q + 3)(q - 1)$.

Since S is primitive, it lies in a primitive maximal solvable subgroup M of $\mathrm{GL}_n(q)$. Since $e = 2$, $M = M_6$ (see [36, §8.1] for the definition). By [36, Proposition 8.2.1],

$$M = M_2 \otimes M_i$$

where M_i is defined in Remark 3.13 for $i = 2, 3, 4$. Recall also the values of $c_r(M_i)$ for $r = 2, 3$ from Remark 3.13. Now, since $M_2 \otimes I_2$ and $I_2 \otimes M_i$ contain $Z(\mathrm{GL}_4(q))$, we deduce that $c_2(M) = (q + 3)\cdot 10(q - 1)$ and $c_3(M) = (3, q + 1)\cdot 9(q - 1)$. So there are $9(3, q + 1)$ elements g of $M/Z(\mathrm{GL}_4(q))$ such that $g^3 = 1$ and, therefore, $A_3 := 9(3, q + 1) - 1$ elements of order 3. Similarly, there are $A_2 := 10(q + 3) - 1$ elements of order 2 in $M/Z(\mathrm{GL}_4(q))$.

If $r = 2$, then x is semisimple and $|x^G| > B_2 := (1/4)q^4$ by [9, Table 3.8]. So, by Lemma 2.18,

$$Q_2 \le A_2^3/B_2^2 = \frac{(10(q + 3) - 1)^3}{((1/4)q^4)^2}.$$

If $r = 2$, then $|x^G| > B_3 := (1/8)(q/(q + 1))q^3$ by Lemma 2.25. So, by Lemma 2.18,

$$Q_3 \le A_3^3/B_3^2 = \frac{(9(3, q + 1) - 1)^3}{((1/8)(q/(q + 1))q^3)^2}.$$

Computations show that $Q(G, 3) \le Q_1+Q_2+Q_3 < 1$ for $q > 11$. If $q \le 11$, then $b_S(\mathrm{Sp}_4(q)) \le 3$ is established by computation. □

3.2. **Imprimitive irreducible subgroups.** We commence by obtaining a result about the group of monomial matrices in $\mathrm{GU}_n(q)$. Recall that by default we assume that $\mathrm{GU}_n(q)$ is $\mathrm{GU}_n(q, I_n)$, the general unitary group with respect to an orthonormal basis of V. We combine this result with those of the previous section to obtain an upper bound to $b_S(S\cdot(\mathrm{SL}_n(q^{\mathbf{u}})\cap G))$ for those maximal solvable subgroups S of $G \in \{\mathrm{GU}_n(q), \mathrm{GSp}_n(q)\}$ which are not quasi-primitive.

For $a \in \mathbb{F}_{q^{\mathbf{u}}}^*$ let $B(n, a)$ and $C(n, a)$ be the following $n \times n$ matrices:

$$B(n,a) = \begin{pmatrix} a & a & 0 & 0 & 0 & \dots & 0 \\ a^2 & -a^2 & a & 0 & 0 & \dots & 0 \\ a^3 & -a^3 & -a^2 & a & 0 & \dots & \\ & & & \ddots & & \ddots & \\ a^{(n-2)} & -a^{(n-2)} & -a^{(n-3)} & \dots & -a^2 & a & 0 \\ a^{(n-1)} & -a^{(n-1)} & -a^{(n-2)} & \dots & -a^3 & -a^2 & a \\ a^{(n-1)} & -a^{(n-1)} & -a^{(n-2)} & \dots & -a^3 & -a^2 & -a \end{pmatrix};$$

$$C(n,a) = \begin{pmatrix} \sqrt{a^2+1} & a & 0 & 0 & \dots & 0 \\ \sqrt{a^2+1} & (a+a^{-1}) & a^{-1} & 0 & \dots & 0 \\ \sqrt{a^2+1} & (a+a^{-1}) & (a+a^{-1}) & a & \dots & 0 \\ & & \vdots & \vdots & \vdots & \\ \sqrt{a^2+1} & (a+a^{-1}) & \dots & (a+a^{-1}) & (a+a^{-1}) & a^{(-1)^n} \\ \alpha & \delta & \dots & \delta & \delta & \delta \end{pmatrix}.$$

Here $n \geq 3$ for $B(n,a)$ and $C(n,a)$. Denote by $B'(n,a)$ the matrix $B(n,a)\pi$, where π is the permutation matrix for the permutation $(1,n)(2,n-1)\dots([n/2],[n/2+3/2])$. If n is even, then $\alpha = a$ and $\delta = \sqrt{a^2+1}$ in $C(n,a)$. If n is odd, then $\alpha = 1$ and $\delta = a^{-1}\sqrt{a^2+1}$.

Lemma 3.14. *Let $M_n(q)$ be the group of all monomial matrices in $\mathrm{GL}_n(q)$ and $\mathrm{MU}_n(q) := M_n(q^2) \cap \mathrm{GU}_n(q)$.*

(1) *If q is odd and $a \in \mathbb{F}_{q^2}$ satisfies $a^{q+1} = 2^{-1}$, then*

$$\mathrm{MU}_n(q) \cap \mathrm{MU}_n(q)^{B(n,a)} \cap \mathrm{MU}_n(q)^{B'(n,a)} \leq Z(\mathrm{GU}_n(q))$$

for $n \geq 3$.

(2) *If q is even and $1 \neq a \in \mathbb{F}_q^*$ (so $q > 2$), then*

$$\mathrm{MU}_n(q) \cap \mathrm{MU}_n(q)^{C(n,a)} \leq Z(\mathrm{GU}_n(q))$$

for $n \geq 3$.

(3) *If $q \geq 4$, then $b_{\mathrm{MU}_2(q)}(\mathrm{GU}_2(q)) \leq 3$.*

(4) *If $n = 4$, then $b_{\mathrm{MU}_n(2)}(\mathrm{GU}_n(2)) = 4$. If $n > 4$, then $b_{\mathrm{MU}_n(2)}(\mathrm{GU}_n(2)) \leq 3$.*

Proof. (1) Since q is odd, there always exists $a \in \mathbb{F}_{q^2}$ such that $a^{q+1} = 2^{-1}$. Indeed, let η be a generator of $\mathbb{F}_{q^2}^*$ and $\theta = \eta^{q+1}$, so θ is a generator of $\mathbb{F}_q^*$. Thus, $\theta^k = 2^{-1}$ for some integer k. Therefore, $(\eta^k)^{q+1} = 2^{-1}$. It is routine to check that $B(n,a)$ and $B'(n,a)$ lie in $\mathrm{GU}_n(q)$ for such a.

Consider $g \in \mathrm{MU}_n(q) \cap \mathrm{MU}_n(q)^{B(n,a)}$, so

$$g = \mathrm{diag}(g_1, \dots, g_n)r,$$

where $r \in \mathrm{Sym}(n)$ and $g_i \in \mathbb{F}_{q^2}^*$.

Let $\beta = \{v_1, \dots, v_n\}$ be the orthonormal basis of V such that $\mathrm{GU}_n(q) = \mathrm{GU}_n(q, \mathbf{f}_\beta)$. Since g is monomial, it stabilises the decomposition

$$\langle v_1 \rangle \oplus \dots \oplus \langle v_n \rangle.$$

Since $g \in \mathrm{GU}_n(q)^{B(n,a)}$, it stabilises the decomposition

$$\langle (v_1)B(n,a) \rangle \oplus \dots \oplus \langle (v_n)B(n,a) \rangle.$$

We write w_i for $(v_i)B(n,a)$. Notice that

$$w_i = \begin{cases} av_1 + av_2 & \text{if } i = 1; \\ a^i v_1 - a^i v_2 - a^{i-1}v_3 \dots - a^2 v_i + av_{i+1} & \text{if } 1 < i < n; \\ a^{n-1}v_1 - a^{n-1}v_2 - a^{n-2}v_3 \dots - a^2 v_{n-1} - av_n & \text{if } i = n. \end{cases}$$

Since g is monomial, w_i and $(w_i)g$ have the same number of non-zero entries (which is $i+1$ for $i \neq n$ and n for $i = n$) in the decomposition with respect to β. Therefore, $(w_i)g \in \langle w_i \rangle$ for $i < n-1$, so r must fix $\{1, 2\}$ and points $3, \dots, n$. Thus, $(w_{n-1})g$ is either δw_{n-1} or δw_n for some $\delta \in \mathbb{F}_{q^2}$. If r fixes the point 1, then $\delta = g_1 = g_2 = \ldots = g_{n-1} = \pm g_n$; if $(1)r = 2$, then $\delta = -g_1 = -g_2 = g_3 = \ldots = g_{n-1} = \pm g_n$. It is easy to see that $\mathrm{MU}_n(q) \cap \mathrm{MU}_n(q)^{B(n,q)}$ lies in

$$\{\mathrm{diag}((-1)^i\alpha, (-1)^i\alpha, \alpha, \ldots, \alpha, \pm\alpha) \cdot (1,2)^i \mid \alpha \in \mathbb{F}_{q^2}; i \in \{0,1\}\}.$$

Therefore, $\mathrm{MU}_n(q) \cap \mathrm{MU}_n(q)^{B(n,q)\pi} = (\mathrm{MU}_n(q) \cap \mathrm{MU}_n(q)^{B(n,q)})^{\pi}$ lies in

$$\begin{aligned}&\{\mathrm{diag}((-1)^i\alpha, (-1)^i\alpha, \alpha, \ldots, \alpha, \pm\alpha) \cdot (1,2)^i \mid \alpha \in \mathbb{F}_{q^2}; i \in \{0,1\}\}^{\pi} \\ \subseteq &\{\mathrm{diag}(\pm\alpha, \alpha, \ldots, \alpha, (-1)^i\alpha, (-1)^i\alpha) \cdot (n, n-1)^i \mid \alpha \in \mathbb{F}_{q^2}; i \in \{0,1\}\}\end{aligned}$$

and

$$\mathrm{MU}_n(q) \cap \mathrm{MU}_n(q)^{B(n,q)} \cap \mathrm{MU}_n(q)^{B(n,q)\pi} \leq Z(\mathrm{GU}_n(q)).$$

(2) Let $1 \neq a \in \mathbb{F}_q$. Since $\phi : \mathbb{F}_q \to \mathbb{F}_q$ mapping x to x^2 is a Frobenius automorphism of $\mathbb{F}_q$, every element of $\mathbb{F}_q$ has a unique square root in $\mathbb{F}_q$. Therefore, the matrix $C(n,a)$ exists and lies in $\mathrm{GU}_n(q)$.

Suppose that $n \geq 3$ is odd and consider $g \in \mathrm{MU}_n(q) \cap \mathrm{MU}_n(q)^{C(n,a)}$, so

$$g = \mathrm{diag}(g_1, \ldots, g_n)r.$$

Let $\beta = \{v_1, \ldots, v_n\}$ be the orthonormal basis as in (1). Since g is monomial, it stabilises the decomposition

$$\langle v_1 \rangle \oplus \ldots \oplus \langle v_n \rangle.$$

Since $g \in \mathrm{MU}_n(q)^{C(n,a)}$, it stabilises the decomposition

$$\langle (v_1)C(n,a) \rangle \oplus \ldots \oplus \langle (v_n)C(n,a) \rangle.$$

We write w_i for $(v_i)C(n,a)$. Notice that

$$\begin{aligned} w_1 &= \sqrt{a^2+1}v_1 + av_2; \\ w_n &= \alpha v_1 + \delta v_2 + \ldots + \delta v_{n-1} + \delta v_n, \end{aligned}$$

and if $1 < i < n$, then

$$w_i = \begin{cases} \sqrt{a^2+1}v_1 + (a+a^{-1})v_2 + \ldots + (a+a^{-1})v_i + av_{i+1} & \text{if } i \text{ is odd;} \\ \sqrt{a^2+1}v_1 + (a+a^{-1})v_2 + \ldots + (a+a^{-1})v_i + -av_{i+1} & \text{if } i \text{ is even.} \end{cases}$$

Since g is monomial, w_i and $(w_i)g$ have the same number of non-zero entries (which is $i+1$ for $i \neq n$ and n for $i = n$) in the decomposition with respect to β. Therefore, $(w_i)g \in \langle w_i \rangle$ for $i < n-1$, so r must fix $\{1,2\}$ and points $3, \ldots, n$. Assume that $(1)r = 2$, so

$$(w_1)g = g_1\sqrt{a^2+1}v_2 + g_2av_1.$$

Since $(w_1)g \in \langle w_1 \rangle$,

$$g_1\sqrt{a^2+1}v_2 + g_2av_1 = \gamma(\sqrt{a^2+1}v_1 + av_2)$$

for some $\gamma \in \mathbb{F}_{q^2}$. Calculations show that $g_2(g_1)^{-1} = 1 + a^{-2}$. Notice that $g_1^{q+1} = g_2^{q+1} = 1$, since $g \in \mathrm{MU}_n(q)$. Hence $(g_2(g_1)^{-1})^{q+1}$ must be 1. However,

$$(1+a^{-2})^{q+1} = (1+a^{-2})^2 = 1 + a^{-4} \neq 1.$$

So r must fix the points 1 and 2.

Since $(w_{n-2})g \in \langle w_{n-2} \rangle$, we obtain $g_1 = \ldots = g_{n-1}$. Assume that $(w_{n-1})g = \gamma w_n$ for some $\gamma \in \mathbb{F}_{q^2}$. Then $g_1 = \gamma\sqrt{a^2+1}$ and $g_n = \gamma(\sqrt{a^2+1})^{-1}$. Since $(g_i)^{q+1} = 1$ for all $i = 1, \ldots, n$,

$$\gamma^{q+1}(a^2+1) = \gamma^{q+1}(a^2+1)^{-1} = 1.$$

Therefore, a^2+1 must be equal to $(a^2+1)^{-1}$, which is not true since

$$(a^2+1)^2 = a^4+1 \neq 1.$$

Thus, $(w_{n-1})g = \gamma w_{n-1}$ and g is a scalar.

The proof of (2) for even n is analogous to that for odd n.

(3) For $q=5$ the statement is verified by computation. For $q \neq 5$ the statement follows from Lemma 2.13 since $\mathrm{SU}_2(q) \cong \mathrm{SL}_2(q)$.

(4) For $n<7$ the statement is verified by computation. Assume $n \geq 7$. Let a be a generator of $\mathbb{F}_4^*$, so $a^2 = a+1$ and $a^3 = 1$. Let $\beta = \{v_1, \dots, v_n\}$ be the orthonormal basis of V such that $\mathrm{GU}_n(q) = \mathrm{GU}_n(q, \mathbf{f}_\beta)$. Let $E(n,a) \in \mathrm{GU}_n(2)$ be defined as follows. If n is even, then

$$(v_i)E(n,a) = w_i = \begin{cases} v_1 & \text{if } i=1; \\ \sum_{j=2}^n v_j & \text{if } i=2; \\ (a+1)v_i + av_{i+1} + \sum_{j=i+2}^n v_j & \text{if } i \text{ is odd and } 3 \leq i \leq n; \\ av_i + (a+1)v_{i+1} + \sum_{j=i+2}^n v_j & \text{if } i \text{ is even and } 3 \leq i \leq n. \end{cases}$$

If n is odd, then

$$(v_i)E(n,a) = w_i = \begin{cases} \sum_{j=1}^n v_j & \text{if } i=1; \\ (a+1)v_i + av_{i+1} + \sum_{j=i+2}^n v_j & \text{if } i \text{ is even and } 2 \leq i \leq n; \\ av_i + (a+1)v_{i+1} + \sum_{j=i+2}^n v_j & \text{if } i \text{ is odd and } 2 \leq i \leq n. \end{cases}$$

For example, $E(8,a)$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & a+1 & a & 1 & 1 & 1 & 1 & 1 \\ 0 & a & a+1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & a+1 & a & 1 & 1 & 1 \\ 0 & 0 & 0 & a & a+1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & a+1 & a & 1 \\ 0 & 0 & 0 & 0 & 0 & a & a+1 & 1 \end{pmatrix}.$$

We obtain $E(7,a)$ by deleting the first row and the first column in $E(8,a)$. It is routine to verify that $E(n,a) \in \mathrm{GU}_n(2)$.

Let $\pi \in \mathrm{GU}_n(2)$ be the permutation matrix corresponding to the permutation

$$(1,n)(3,n-1)(5,n-2)(6,7,\dots,n-3) \text{ if } n \text{ is even;}$$
$$(1,n)(3,n-1)(5,6,7,\dots,n-3) \text{ if } n \text{ is odd.}$$

We claim that $\mathrm{MU}_n(2) \cap \mathrm{MU}_n(2)^{E(n,a)} \cap \mathrm{MU}_n(2)^{E(n,a)^\pi} \leq Z(\mathrm{GU}_n(2))$. We prove this for even n; the proof is analogous for odd n.

Consider $g \in \mathrm{MU}_n(2) \cap \mathrm{MU}_n(2)^{E(n,a)}$, so

$$g = \mathrm{diag}(g_1, \dots, g_n)r,$$

where $r \in \mathrm{Sym}(n)$ and $g_i \in \mathbb{F}_4^*$. Since g is monomial, it stabilises the decomposition

$$\langle v_1 \rangle \oplus \dots \oplus \langle v_n \rangle.$$

Since $g \in \mathrm{MU}_n(2)^{E(n,a)}$, it stabilises the decomposition

$$\langle w_1 \rangle \oplus \ldots \oplus \langle w_n \rangle.$$

Since g is monomial, w_i and $(w_i)g$ have the same number of non-zero entries in the decomposition with respect to β. Therefore, $(w_i)g \in \langle w_{n-2}, w_{n-1}, w_n \rangle$ for $n-2 \leq i \leq n$, so r must fix $\{n-2, n-1, n\}$. If $i \in \{n-4, n-3\}$, then $(w_i)g \in \langle w_{n-4}, w_{n-3} \rangle$, so r must fix $\{n-4, \ldots, n\}$ and, therefore, $\{n-4, n-3\}$. Continuing this process, we obtain that r fixes

$$\{1\}, \{2, 3\}, \{4, 5\}, \ldots, \{n-4, n-3\}, \{n-2, n-1, n\}. \tag{3.1}$$

Now assume $g \in \mathrm{MU}_n(2) \cap \mathrm{MU}_n(2)^{E(n,a)^\pi}$. The above arguments show that r must fix

$$\{(1)\pi\}, \{(2)\pi, (3)\pi\}, \{(4)\pi, (5)\pi\}, \ldots, \{(n-4)\pi, (n-3)\pi\}, \{(n-2)\pi, (n-1)\pi, (n)\pi\}$$

which are

$$\{n\}, \{2, n-1\}, \{4, n-2\}, \{7, 8\}, \ldots, \{n-3, 6\}, \{1, 3, 4\}.$$

Combining this with (3.1) we obtain that r is a trivial permutation, so g is diagonal.

Observe that $(w_2)g = (0, g_2, g_3, \ldots, g_n)$ with respect to β. Since $(w_2)g \in \langle w_2, w_3, w_4 \rangle$,

$$g_4 = g_5 = \ldots = g_n.$$

So $g = \mathrm{diag}(g_1, g_2, g_3, \lambda, \ldots, \lambda)$ for some $\lambda \in \mathbb{F}_4^*$. Let $u_i = (v_i)E(n,a)^\pi$. Therefore,

$$\begin{aligned} u_2 &= (1, \ldots, 1, 0) \\ u_{n-1} &= (1, a+1, 1, \ldots, 1, a, 0) \\ u_4 &= (1, a, 1, \ldots, 1, a+1, 0) \end{aligned}$$

are the only vectors in $\{u_1, \ldots, u_n\}$ that have $n-1$ non-zero entries in the decomposition with respect to β. Hence $(u_2)g$ lies in $\langle u_2 \rangle$, $\langle u_{n-1} \rangle$, or $\langle u_4 \rangle$ since $g \in \mathrm{MU}_n(2)^{E(n,a)^\pi}$ and stabilises the decomposition

$$\langle u_1 \rangle \oplus \ldots \oplus \langle u_n \rangle.$$

Notice that $(u_2)g = (g_1, g_2, g_3, \lambda, \ldots, \lambda, 0)$ Hence $(u_2)g \in \langle u_2 \rangle$ and $g_1 = g_2 = g_3 = \lambda$. So $g = \lambda I_n \in Z(\mathrm{GU}_n(q))$. Therefore,

$$\mathrm{MU}_n(2) \cap \mathrm{MU}_n(2)^{E(n,a)} \cap \mathrm{MU}_n(2)^{E(n,a)^\pi} \leq Z(\mathrm{GU}_n(2)). \qquad \square$$

Remark 3.15. In Lemma 3.14 each statement of (1)-(4) can be written as

$$S \cap S^{x_1} \cap \ldots \cap S^{x_t} \leq Z(\mathrm{GU}_n(q))$$

for $x_i \in \mathrm{GU}_n(q)$ with suitable S. In each case we can assume $x_i \in \mathrm{SU}_n(q)$. Indeed, if $\det(x_i) \neq 1$, then $a_i = \mathrm{diag}(\det(x_i)^{-1}, 1 \ldots, 1) \in S$ since S is the group of all monomial matrices in $\mathrm{GU}_n(q)$, so $S^{a_i x_i} = S^{x_i}$.

Theorem 3.16. *Let $Z(\mathrm{GU}_m(q)) \leq H \leq \mathrm{GU}_m(q)$. Assume that there exist $a, b \in \mathrm{GU}_m(q)$ such that*

$$H \cap H^a \cap H^b \leq Z(\mathrm{GU}_m(q)).$$

Let $M = (\mathbb{F}_{q^2}^*)^{q-1} \wr \Gamma$ for $\Gamma \leq \mathrm{Sym}(k)$, so M is a subgroup of monomial matrices in $\mathrm{GU}_k(q)$. Assume that there exist $x, y \in \mathrm{GU}_k(q)$ such that

$$M \cap M^x \cap M^y \leq Z(\mathrm{GU}_k(q)).$$

Denote

$$X = I_m \otimes x \qquad A = a \otimes I_k$$
$$Y = I_m \otimes y \qquad B = b \otimes I_k.$$

If $n = mk$ and $S = H \wr \Gamma \leq \mathrm{GU}_n(q)$, then

$$S \cap S^{AX} \cap S^{BY} \leq Z(\mathrm{GU}_n(q)).$$

Proof. Consider $h \in S \cap S^{AX}$, so $h = g^{AX}$ where $g \in S$. Hence

$$g^A = \mathrm{diag}[g_1, \dots, g_k] \cdot \pi,$$

where $g_i \in H^a$ and $\pi = I_m \otimes \pi_1$ for some $\pi_1 \in \mathrm{Sym}(k)$. If

$$x = \begin{pmatrix} x_{11} & \dots & x_{1k} \\ \vdots & & \vdots \\ x_{k1} & \dots & x_{kk} \end{pmatrix}, \text{ then } x^{-1} = \begin{pmatrix} x_{11}^q & \dots & x_{k1}^q \\ \vdots & & \vdots \\ x_{1k}^q & \dots & x_{kk}^q \end{pmatrix},$$

since $x \in \mathrm{GU}_k(q)$, and

$$X = \begin{pmatrix} x_{11}I_m & \dots & x_{1k}I_m \\ \vdots & & \vdots \\ x_{k1}I_m & \dots & x_{kk}I_m \end{pmatrix}.$$

Here $x_{ij} \in \mathbb{F}_{q^2}$. The i-th $(k \times k)$-row of $X^{-1}g^A$ is equal to

$$(x^q_{(1)\pi_1^{-1}i}g_{(1)\pi_1^{-1}}, \dots, x^q_{(k)\pi_1^{-1}i}g_{(k)\pi_1^{-1}}). \tag{3.2}$$

Let j be such that the (i,j)-th $(m \times m)$-block of h is not zero (there is only one such j for given i since $h \in S$). Consider the system of linear equations with variables $Z_1, \dots, Z_k \in H^a$

$$\begin{aligned} &x_{11}Z_1 + x_{21}Z_2 + \dots + x_{k1}Z_k = 0 \\ &\vdots \\ &\underline{x_{1j}Z_1 + x_{2j}Z_2 + \dots + x_{kj}Z_k = 0} \\ &\vdots \\ &x_{1k}Z_1 + x_{2k}Z_2 + \dots + x_{kk}Z_k = 0, \end{aligned} \tag{3.3}$$

where we exclude the (underlined) j-th equation. Thus, (3.3) consist of $k-1$ linearly independent equations. If we fix Z_k to be some matrix from $\mathrm{GL}_n(q^2)$, then Z_i for $i = 1, \dots, k-1$ are determined uniquely. It is routine to check that

$$(x^q_{1j}D, \dots, x^q_{kj}D) \text{ where } D \in \mathrm{GL}_n(q^2)$$

is a solution for the system (3.3).

Notice that the row (3.2) must be a solution of (3.3), since $X^{-1}g^AX = h \in S$. Therefore, by fixing Z_k to be $x^q_{kj}D_i := x^q_{(k)\pi_1^{-1}i}g_{(k)\pi_1^{-1}}$, we obtain

$$(x^q_{(1)\pi_1^{-1}i}g_{(1)\pi_1^{-1}}, \dots, x^q_{(k)\pi_1^{-1}i}g_{(k)\pi_1^{-1}}) = (x^q_{1j}D_i, \dots, x^q_{kj}D_i)$$

for some $D_i \in \alpha H^a$, $\alpha \in \mathbb{F}^*_{q^2}$, since $g_i \in H^a$. Thus,

$$h = \mathrm{diag}[h_1, \dots, h_k] \cdot \sigma$$

where $h_i = D_i$. Therefore, $\alpha^{q+1} = 1$ and $D_i \in H^a$, since $h_i \in \mathrm{GU}_m(q)$. So $h_i \in H \cap H^a$ and $\sigma \in I_m \otimes \mathrm{Sym}(k)$.

Assume that $h \in S \cap S^{BY}$, so $h = (g')^{BY}$ for $g' \in S$. The same argument as above shows that

$$h = \mathrm{diag}[h_1, \dots, h_k] \cdot \sigma$$

where $h_i \in H \cap H^b$ and $\sigma \in I_m \otimes \mathrm{Sym}(k)$.

Therefore, if $h \in S \cap S^{AX} \cap S^{BY}$ then $h_i = \lambda_i I_m \in H \cap H^a \cap H^b$ for some $\lambda_i \in \mathbb{F}_{q^2}^*$ with $\lambda_i^{q+1} = 1$. So $g^A, g'^B \in I_m \otimes M$ and

$$h \in I_m \otimes (M \cap M^x \cap M^y) \leq Z(\mathrm{GU}_n(q)). \qquad \square$$

Remark 3.17. If $a, b \in \mathrm{SU}_m(q)$ and $x, y \in \mathrm{SU}_k(q)$ in Theorem 3.16, then $AX, BY \in \mathrm{SU}_n(q)$, since $AX = a \otimes x$ and $BY = b \otimes y$.

Lemma 3.18. *Let $Z(\mathrm{GU}_m(q)) \leq H_i \leq \mathrm{GU}_m(q)$ for $i \in \{1, \dots, k\}$ such that $\mathrm{GU}_k(q)$ is not solvable. Assume that there exist $a_i, b_i \in \mathrm{GU}_m(q)$ such that*

$$H_i \cap H_i^{a_i} \cap H_i^{b_i} \leq Z(\mathrm{GU}_m(q)).$$

Let $H = \{\mathrm{diag}[h_1, \dots, h_k] \mid h_i \in H_i\} \leq \mathrm{GU}_{mk}(q)$, so $H = H_1 \times \ldots \times H_k$. Then there exist $A, B \in \mathrm{GU}_{mk}(q)$ such that $H \cap H^A \cap H^B \leq Z(\mathrm{GU}_{mk}(q))$.

Proof. Let D be the subgroup of all diagonal matrices in $\mathrm{GU}_k(q) = \mathrm{GU}_k(q, I_k)$. By Theorem 2.10, there exists $x \in \mathrm{GU}_k(q)$ such that $D \cap D^x \leq Z(\mathrm{GU}_k(q))$. Let $X = (I_m \otimes x)$, $A = \mathrm{diag}[a_1, \dots, a_k]X$ and $B = \mathrm{diag}[b_1, \dots, b_k]$.

Consider $h \in H \cap H^A$, so $h = g^A$ where $g \in H$. Hence

$$g^A = \mathrm{diag}[g_1, \dots, g_k]^X$$

where $g_i \in H_i^{i_a}$. The same arguments as in the proof of Theorem 3.16 below (3.2) (with $j = i$ and π trivial) show that $h = \mathrm{diag}[h_1, \dots, h_k]$ with $h_i \in H_i \cap H_i^{a_i}$. Assume that, in addition, $h \in \cap H^B$; then, clearly, $h_i \in H_i \cap H_i^{a_i} \cap H_i^{b_i} \leq Z(\mathrm{GU}_m(q))$. Therefore, $h^{X^{-1}} = g^{\mathrm{diag}[a_1, \dots, g_k]} \in I_m \otimes D$, so $h \in I_m \otimes (D \cap D^x) \leq Z(\mathrm{GU}_{mk}(q))$. $\square$

Lemma 3.19. *Let $k \in \{2, 4, 6, 8\}$. Let H be an irreducible subgroup of $GU(V)$ that stabilises the decomposition*

$$V = V_1 \oplus \ldots \oplus V_k$$

as in (2) *of Lemma* 2.6*, so each V_i is totally isotropic and $\dim V_i = m$, where $n = km$. Denote $\mathrm{Stab}_H(V_1)|_{V_1} \leq \mathrm{GL}(V_1)$ by H_1. If there exist $a, b \in \mathrm{GL}(V_1)$ (respectively $\mathrm{SL}(V_1)$) such that*

$$H_1 \cap H_1^a \cap H_1^b \leq Z(\mathrm{GL}(V_1)),$$

then there exist $A, B \in GU(V)$ (respectively $SU(V)$) such that

$$H \cap H^A \cap H^B \leq Z(GU(V)).$$

Proof. Let $\alpha \in \mathbb{F}_{q^2}^*$ be such that $\alpha + \alpha^q = 0$. Such α always exists. Indeed, if q is even, then α can be an arbitrary element of $\mathbb{F}_q^*$. Assume that q is odd and η is a generator of $\mathbb{F}_{q^2}^*$, so $\eta^{(q^2-1)/2} = -1$ is the unique element of order 2 in $\mathbb{F}_{q^2}^*$. Let $\alpha = \eta^{(q+1)/2}$, therefore, $\alpha^{q-1} = -1$ and $\alpha^q = -\alpha$, so $\alpha + \alpha^q = 0$.

Assume $k=2$. Let β be a basis as in (2.3). Since V_1 is totally isotropic, we can assume that $V_1=\langle f_1,\dots,f_m\rangle$ by Lemma 2.4. Every $v\in V$ has a unique decomposition $v=v_1+v_2$, $v_i\in V_i$. Define the projection operators $\pi_i : V\to V_i$ by $(v)\pi_i=v_i$ for $i=1,2$. Notice that

$$(f_i,e_j)=(f_i,(e_j)\pi_1+(e_j)\pi_2)=(f_i,(e_j)\pi_2)$$

since V_1 is totally isotropic. Also $((e_i)\pi_2,(e_j)\pi_2)=0$ since V_2 is totally isotropic. Therefore, the form $\mathbf{f}$ has matrix $\mathbf{f}_{\beta_1}=J_{2m}$ with respect to the basis $\beta_1=\{f_1,\dots,f_m,(e_1)\pi_2,\dots,(e_m)\pi_2\}$. In other words, we can assume that

$$V_2=\langle e_1,\dots,e_m\rangle.$$

Therefore, applying the above argument to each $U_i=V_{2i-1}\oplus V_{2i}$ for $i\in\{1,2,3\}$, we obtain a basis

$$\beta=\{f_{11},\dots,f_{1m},e_{11},\dots,e_{1m},\dots,f_{(k/2)1},\dots,f_{(k/2)m},e_{(k/2)1},\dots,e_{(k/2)m}\}$$

of V such that $\mathbf{f}_\beta=J_{2m}\otimes I_{k/2}$ and $V_{2i-1}=\langle f_{i1},\dots,f_{im}\rangle$, $V_{2i}=\langle e_{i1},\dots,e_{im}\rangle$.

Recall that $g^\dagger=(\overline{g}^\top)^{-1}$ for $g\in \mathrm{GU}_n(q)$. For $x\in \mathrm{GL}_m(q^2)$, denote by $X(k,x)$ the initial $(k\times k)$-submatrix of the matrix

$$X(x)=\left(\begin{array}{cc|cc|cc|cc}
\alpha x & x & \alpha x & x & 0 & 0 & 0 & 0\\
0 & (\alpha x)^\dagger & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
0 & -x & 0 & x & 0 & 0 & 0 & 0\\
0 & 0 & x^\dagger & -(\alpha x)^\dagger & x^\dagger & -(\alpha x)^\dagger & 0 & 0\\
\hline
0 & 0 & 0 & 0 & \alpha x & x & 0 & 0\\
0 & (\alpha x)^\dagger & 0 & -(\alpha x)^\dagger & 0 & (\alpha x)^\dagger & x^\dagger & -(\alpha x)^\dagger\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \alpha x & x\\
0 & 0 & 0 & 0 & x^\dagger & -(\alpha x)^\dagger & 0 & (\alpha x)^\dagger
\end{array}\right).$$

It is routine to check that $X(k,x)\in \mathrm{GU}_n(q,\mathbf{f}_\beta)$. If $g\in H_\beta\cap H_\beta^{X(k,a)}$, then g stabilises both

$$V=V_1\oplus\ldots\oplus V_k, \tag{3.4}$$

and

$$V=(V_1)X(k,a)\oplus\ldots\oplus(V_k)X(k,a). \tag{3.5}$$

Let $k=8$ and $v\in V$. Since g stabilises (3.4), $(v)g$ and v have the same number of non-zero projections on the V_i. Hence g stabilises $(V_2)X(k,a)$ and V_2 because $(V_2)X(k,a)$ is the only subspace in (3.5) which has only one non-zero projection on the V_i. Therefore, g stabilises V_1 and $(V_2)X(k,a)$ because they are the only subspaces which are not orthogonal to V_2 and $(V_2)X(k,a)$ respectively in decompositions (3.4) and (3.5). Since g stabilises V_2, it stabilises $(V_3)X(k,a)$, so g also stabilises V_3, V_4 and $(V_4)X(k,a)$. Since g stabilises $(V_4)X(k,a)$, it stabilises $V_5\oplus V_6$, so it stabilises $(V_5)X(k,a)$ and $(V_6)X(k,a)$. Now it is easy to see that g must stabilise V_5 and V_6. Since g stabilises $(V_7)X(k,a)\oplus(V_8)X(k,a)$, it stabilises $(V_8)X(k,a)$, so g stabilises V_8 and V_7. Therefore, g stabilises all subspaces in (3.4) and (3.5), so $g=\mathrm{diag}[g_1,g_1^\dagger,\dots,g_{k/2},g_{k/2}^\dagger]$ with $g_i\in H_1$.

Since $X(x)^{-1} = \mathbf{f}_\beta \overline{X(x)}^\top \mathbf{f}_\beta$,

$$X(x)^{-1} = \left(\begin{array}{cc|cc|cc|cc} (\alpha x)^{-1} & \overline{x}^\top & 0 & -\overline{x}^\top & 0 & 0 & 0 & 0 \\ 0 & (\overline{\alpha x})^\top & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \overline{x}^\top & -(\alpha x)^{-1} & \overline{x}^\top & -(\alpha x)^{-1} & 0 & 0 & 0 \\ 0 & (\overline{\alpha x})^\top & x^{-1} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -(\alpha x)^{-1} & 0 & (\alpha x)^{-1} & \overline{x}^\top & -(\alpha x)^{-1} & 0 \\ 0 & 0 & x^{-1} & 0 & 0 & (\overline{\alpha x})^\top & x^{-1} & 0 \\ \hline 0 & 0 & 0 & 0 & -(\alpha x)^{-1} & 0 & (\alpha x)^{-1} & \overline{x}^\top \\ 0 & 0 & 0 & 0 & x^{-1} & 0 & 0 & (\overline{\alpha x})^\top \end{array}\right).$$

A similar argument to the above shows that if $h = g^{X(k,a)^{-1}} \in H_\beta \cap H_\beta^{X(k,a)^{-1}}$, then $h = \mathrm{diag}[h_1, h_1^\dagger, \ldots, h_{k/2}, h_{k/2}^\dagger]$ with $h_i \in H_1$. Calculations show that if the equation $h^{X(k,a)} = g$ holds, then

$$\begin{cases} g_i & = g(2i-1, 2i-1) = h_i^a \text{ for } i \in \{1,2,3,4\} \\ 0 & = g(1,2) = (\alpha a)^{-1} h_1 a + \overline{a}^\top h_1^\dagger (\alpha a)^\dagger = \alpha^{-1}(h_1^a - (h_1^a)^\dagger); \\ 0 & = g(1,3) = (\alpha a)^{-1} h_1 (\alpha a) - \overline{a}^\top h_2^\dagger a^\dagger = h_1^a - (h_2^a)^\dagger; \\ 0 & = g(1,5) = -\overline{x}^\top h_2^\dagger a^\dagger + (\alpha a)^{-1} h_3 (\alpha a) = h_3^a - (h_2^a)^\dagger; \\ 0 & = g(8,5) = a^{-1} h_3 (\alpha a) + (\overline{\alpha a})^\top h_4^* a^\dagger = \alpha (h_3^a - (h_4^a)^\dagger). \end{cases}$$

So $h_1^a = g_1 = g_1^\dagger \ldots = g_{k/2} = g_{k/2}^\dagger$ and $g_1 \in H_1 \cap H_1^a$.

If $g \in H_\beta \cap H_\beta^{X(k,a)} \cap H_\beta^{X(k,b)}$, then the same argument with a replaced by b shows that $g = \mathrm{diag}[g_1, \ldots, g_1]$ and $g_1 \in H_1 \cap H_1^a \cap H_1^b$, so $g \in Z(\mathrm{GU}_{2m}(q, \mathbf{f}_\beta))$.

The proof for $k \in \{2,4,6\}$ is analogous.

Calculations show that if $x \in \mathrm{SL}_m(q)$, then

$$\det(X(k,x)) = \begin{cases} 1 & \text{for } k = 4, 8; \\ (-1)^m & \text{for } k = 2, 6. \end{cases}$$

Consider $A = \mathrm{diag}[\alpha I_m, \alpha^\dagger I_m, I_m, \ldots, I_m] \in \mathrm{GU}_n(q, \mathbf{f}_\beta)$. Notice that $\det(A) = (-1)^m$. Repeating the arguments above, one can show that

$$H_\beta \cap H_\beta^{AX(k,a)} \cap H_\beta^{AX(k,b)} \leq Z(\mathrm{GU}_n(q, \mathbf{f}_\beta)).$$

Notice that $X(k,a), X(k,b) \in \mathrm{SU}_n(q, \mathbf{f}_\beta)$ for $k = 4, 8$ and $AX(k,a), AX(k,b) \in \mathrm{SU}_n(q, \mathbf{f}_\beta)$ for $k = 2, 6$. □

Lemma 3.20. *Let $n = mk$ for integers $m \geq 3$ and $k \geq 2$.*

(1) *If $S = \mathrm{GU}_2(2) \wr \mathrm{Sym}(k)$, then $b_S(S \cdot \mathrm{SU}_{2k}(2)) \leq 3$.*
(2) *If $S = \mathrm{GU}_2(q) \wr \mathrm{Sym}(k)$, $q \in \{3,5\}$, then $b_S(S \cdot \mathrm{SU}_{2k}(3)) \leq 3$.*
(3) *If $S = \mathrm{GU}_3(2) \wr \mathrm{Sym}(k)$, then $b_S(S \cdot \mathrm{SU}_{3k}(2)) \leq 3$.*
(4) *Let N be a quasi-primitive maximal solvable subgroup of $\mathrm{GU}_m(2)$. If $S = N \wr \mathrm{Sym}(k)$ with $k \in \{2,3,4\}$, then $b_S(S \cdot \mathrm{SU}_{km}(2)) \leq 3$.*
(5) *Let N be a quasi-primitive maximal solvable subgroup of $\mathrm{GU}_m(3)$. If $S = N \wr \mathrm{Sym}(2)$, then $b_S(S \cdot \mathrm{SU}_{2m}(3)) \leq 3$.*

Proof. Notice that $S \cdot \mathrm{SU}_n(q) = \mathrm{GU}_n(q)$ for (1) – (2).

(1) Notice that $\lambda^{q+1} = 1$ for $q = 2$ and $\lambda \in \mathbb{F}_{q^2}^*$. Therefore, since a row v of a matrix in $\mathrm{GU}_n(q)$ satisfies $\mathbf{f}(v,v) = 1$, every matrix in $\mathrm{GU}_2(2)$ is monomial. Thus,

$$S = \mathrm{GU}_2(2) \wr \mathrm{Sym}(k) \leq \mathrm{MU}_{2k}(q)$$

and the statement for $k > 2$ follows by (4) of Lemma 3.14. The case $k = 2$ is verified by computation.

(2) For $k \in \{2, 3\}$ we verify the statement by computation, so assume $k \geq 4$. Let $\beta = \{v_{11}, v_{12}, v_{21}, v_{22}, \dots, v_{k1}, v_{k2}\}$ be an orthonormal basis of V such that S stabilises the decomposition $V_1 \oplus \ldots \oplus V_k$ with $V_i = \langle v_{i1}, v_{i2}\rangle$. Define a basis $\beta_1 = \{w_{11}, w_{12}, w_{21}, w_{22}, \dots, w_{k1}, w_{k2}\}$ by the following rule:

$$(w_{11}, w_{21}, \dots, w_{k1}) = (v_{12}, v_{21}, \dots, v_{k1})B(k, a);$$
$$(w_{12}, w_{22}, \dots, w_{k2}) = (v_{22}, v_{32}, \dots, v_{k2}, v_{12})B(k, a).$$

Here a and $B(k, a)$ are as in (1) of Lemma 3.14. Denote the change-of-basis matrix from β_1 to β by y. For example, if $k = 4$, then

$$y = \left(\begin{array}{cc|cc|cc|cc}
a & & a & & & & & \\
 & & & a & & a & & \\
\hline
a^2 & & -a^2 & & a & & & \\
 & & & a^2 & & -a^2 & & a \\
\hline
a^3 & & -a^3 & & -a^2 & & a & \\
 & a & & a^3 & & -a^3 & & -a^2 \\
\hline
a^3 & & -a^3 & & -a^2 & & -a & \\
 & -a & & a^3 & & -a^3 & & -a^2
\end{array}\right).$$

We use blanks instead of zeroes in the matrix. It is routine to verify that β_1 is orthonormal, so $y \in \mathrm{GU}_{2k}(3)$. Observe $g \in S \cap S^y$ stabilises the decompositions

$$V_1 \oplus \ldots \oplus V_k \text{ and } W_1 \oplus \ldots \oplus W_k,$$

where $W_i = \langle w_{i1}, w_{i2}\rangle$.

Notice that w_{11} has non-zero entries only in two V_i-s, so $(w_{11})g$ also must have non-zero entries only in two V_i-s. It is easy to see that a vector from W_j for $j > 1$ has non-zero entries in at least three V_i-s. Thus, g stabilises W_1. The same argument shows that g must stabilise W_i for $i = 1, \dots, k-2$. Notice that $(w_{ij})g$ lies either in $\langle w_{i1}\rangle$ or $\langle w_{i2}\rangle$ for $i = 1, \dots, k-2$, since otherwise it would have non-zero entries in more V_i-s than w_{ij}.

Assume that $(w_{11})g \in \langle w_{12}\rangle$. Therefore, $(w_{12})g \in \langle w_{11}\rangle$, so either

$$(V_1)g = V_2, (V_2)g = V_3, (V_3)g = V_1,$$

or

$$(V_1)g = V_3, (V_2)g = V_2, (V_3)g = V_1.$$

In both cases $(w_{22})g$ cannot lie in either $\langle w_{21}\rangle$ or $\langle w_{22}\rangle$, which is a contradiction. So g stabilises $\langle w_{11}\rangle$ and, therefore, it stabilises $\langle w_{12}\rangle$, since $\langle w_{12}\rangle$ is the orthogonal complement of $\langle w_{11}\rangle$ in W_1. Therefore, g stabilises V_1, V_2 and V_3. The same argument shows that g stabilises $V_1, \dots, V_{k-1}$, so g stabilises V_k as well. Thus, g stabilises

$$\langle w_{11}\rangle, \langle w_{12}\rangle, \dots, \langle w_{(k-2)1}\rangle, \langle w_{(k-2)2}\rangle,$$

which implies that g stabilises $\langle v_{11}\rangle, \langle v_{12}\rangle, \dots, \langle v_{k1}\rangle, \langle v_{k2}\rangle$. So

$$g = \mathrm{diag}(g_{11}, g_{12}, \dots, g_{k1}, g_{k2}).$$

Since $(w_{11})g \in \langle w_{11}\rangle$, $g_{11} = g_{21}$. Applying the same argument to all w_{ij} for $i = 1, \dots, k-2$ and $j = 1, 2$ we obtain that g is scalar.

(3) For $k \leq 3$ the statement is verified by computation, so assume $k \geq 4$. Fix

$$\beta = \{v_{11}, v_{12}, v_{13}, v_{21}, \dots, v_{k3}\}$$

to be the initial orthonormal basis, so $g \in S$ stabilises the decomposition

$$V = V_1 \oplus \ldots \oplus V_k, \tag{3.6}$$

where $V_i = \langle v_{i1}, v_{i2}, v_{i3}\rangle$. Let x be the permutation matrix for the permutation $(1, 2, \dots, n)$ where $n = 3k$. Consider $g \in S \cap S^x$. We claim that g is monomial. Indeed, since $g \in S^x$, it stabilises the decomposition

$$V = (V_1)x \oplus \ldots \oplus (V_k)x = \langle v_{12}, v_{13}, v_{21}\rangle \oplus \ldots \oplus \langle v_{k2}, v_{k3}, v_{11}\rangle, \tag{3.7}$$

so it permutes subspaces $\langle v_{11}\rangle, \langle v_{21}\rangle, \dots, \langle v_{k1}\rangle$ and $\langle v_{12}, v_{13}\rangle, \langle v_{22}, v_{23}\rangle, \dots, \langle v_{k2}, v_{k3}\rangle$. Thus, g consists of (1×1) and (2×2) blocks which lie in $\mathrm{GU}_1(2)$ and $\mathrm{GU}_2(2) = \mathrm{MU}_2(2)$, respectively.

Define a basis $\beta_1 = \{w_{11}, w_{12}, w_{13}, w_{21}, \dots, w_{k3}\}$ as follows:

$$\begin{aligned}
w_{11} &= (\sum_{i=1}^{k-1}\sum_{j=1}^{3} v_{ij}) + v_{k1} + ((1 + (-1)^k)/2)v_{k2}\\
w_{12} &= v_{11} + v_{12} + v_{k3}\\
w_{13} &= v_{11} + v_{21} + v_{k3}\\
w_{s1} &= v_{11} + v_{(s-1)3} + v_{k3}\\
w_{s2} &= v_{11} + v_{s2} + v_{k3}\\
w_{s3} &= v_{11} + v_{(s+1)1} + v_{k3}\\
w_{k1} &= v_{11} + v_{(k-1)3} + v_{k3}\\
w_{k2} &= v_{11} + v_{k2} + v_{k3}\\
w_{k3} &= ((1 + (-1)^k)/2)v_{12} + v_{13} + (\sum_{i=2}^{k}\sum_{j=1}^{3} v_{ij}).
\end{aligned}$$

Here $1 < s < k$. Denote the change-of-basis matrix from β_1 to β by y. For example, if $k = 3$, then

$$y = \left(\begin{array}{ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1\\
\hline
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1\\
\hline
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right).$$

It is routine to verify that β_1 is orthonormal, so $y \in \mathrm{GU}_{3k}(2)$. If $g \in S \cap S^x \cap S^y$, then g stabilises decompositions (3.6), (3.7) and

$$V = W_1 \oplus \ldots \oplus W_k,$$

where $W_i = \langle w_{i1}, w_{i2}, w_{i3}\rangle$. Since g is monomial, w_{ij} and $(w_{ij})g$ have the same number of non-zero entries in the decomposition with respect to β. Therefore, $(w_{11})g$ can lie either in W_1 or in W_k. Assume that $(w_{11})g \in W_k$, so $(W_1)g = W_k$. If a vector in W_k has the same number of non-zero entries in the decomposition with respect to β as w_{11}, then its first $1 + ((1 + (-1)^k)/2)$ entries are zero. So g must permute subspace $\langle v_{11}, v_{12}\rangle$ with $\langle v_{k2}, v_{k3}\rangle$ for k odd (respectively $\langle v_{11}\rangle$ with $\langle v_{k3}\rangle$ for k even) which contradicts the fact that g stabilises decompositions (3.6) and (3.7). Therefore, g stabilises W_1 and $\langle w_{11}\rangle$ in particular. It is easy to see now that g stabilises $\langle w_{12}\rangle$ and $\langle w_{13}\rangle$, since g stabilises (3.6). Thus, g stabilises V_1, V_k and V_2, so it stabilises $\langle v_{11}\rangle$, $\langle v_{12}\rangle$, $\langle v_{13}\rangle$, $\langle v_{21}\rangle$, $\langle v_{k3}\rangle$. Using the same argument, we obtain that g is diagonal. Since g stabilises $\langle w_{11}\rangle$ and $\langle w_{k3}\rangle$, all non-zero entries of g must be equal, so g is scalar.

(4)-(5) For $n < 12$ we verify the statement by computation. For larger n we prove the statement by checking (2.11) with $c = 3$ for the elements of prime order of $H \leq \mathrm{PGU}_n(q)$, where H and G are the images of S and $S \cdot \mathrm{SU}_n(q)$ respectively in $\mathrm{PGU}_n(q)$.

Let B be the image in G of the block-diagonal subgroup

$$\mathrm{GU}_m(q) \times \ldots \times \mathrm{GU}_m(q) \leq \mathrm{GU}_{km}(q).$$

Let $x \in H$ have prime order.

If ($|x| > 3$ for $k \in \{3, 4\}$) or ($|x| > 2$ for $k = 2$), then $x^G \cap H \subseteq B$. In this case there exists a preimage $\hat{x} = \mathrm{diag}[\hat{x_1}, \ldots, \hat{x_k}]$ of x in $\mathrm{GU}_{km}(q)$ such that $\hat{x_i} \in N$. If $x_i \neq 1$, then $\nu(x_i) \geq m/4$ by Lemma 3.9. We can assume that x is such that the number $l(x)$ of x_i not equal to 1 is maximal for elements in $x^G \cap H$. Therefore,

$$\nu(x) \geq (1/4)l(x)m \text{ and } |x^G \cap H| \leq \tbinom{k}{l(x)}|N|^{l(x)}.$$

These bounds together with bounds from Lemma 3.8 for $|N|$ and (2.12) for $|x^G|$ are sufficient to show (2.11) holds for $mk \geq 12$.

Now consider the case where $x^G \cap H$ is not a subset of B. For such x, we use the bounds for $|x^G|$ and $|x^G \cap H|$ given in [10, Propositions 2.5 and 2.6]. These propositions give corresponding bounds when $H = (\mathrm{GU}_m(q) \wr \mathrm{Sym}(k))/Z(\mathrm{GU}_n(q))$, so they are applicable in our situation. For $mk \geq 12$, these bounds are sufficient to show that (2.11) holds.

We briefly outline how to extract the corresponding bounds. The proofs of the propositions split into several cases depending on $|x|$, m, k and $|H^1(\sigma, E/E^0)|$ (see [9, Definition 3.5]). Notice that if x is semisimple, then $|H^1(\sigma, E/E^0)| = (|x|, q + 1)$ by [8, Lemma 3.35].

Assume that $k = 2$, so $|x| = 2$. If $q = 2$, then we use the bounds from **Case 2.2** of the proof of [10, Proposition 2.6] for unipotent x. If $q = 3$, then we use bounds from **Case 2.4** of the proof of [10, Proposition 2.5] for semisimple x.

Assume that $k \in \{3, 4\}$, so $q = 2$. If $|x| = 2$, then we use the bounds from **Case 2.2** of the proof of [10, Proposition 2.6] for unipotent x. Let $|x| = 3$. We use bounds from **Case 2.2** (if $|H^1(\sigma, E/E^0)| = 1$) and **Case 2.3** (if $|H^1(\sigma, E/E^0)| = 3$) of the proof of [10, Proposition 2.5]. □

Theorem 3.21. *Let (n, q) be such that $\mathrm{GU}_n(q)$ is not solvable. If S is an irreducible maximal solvable subgroup of $\mathrm{GU}_n(q)$, then either $b_S(S \cdot \mathrm{SU}_n(q)) \leq 3$ or $b_S(S \cdot \mathrm{SU}_n(q)) = 4$ and one of the following holds:*

1. *$(n, q) = (2, 5)$ and S is a completely irreducible subgroup with $S/Z(G)$ isomorphic to $2^2.\mathrm{Sp}_2(2)$;*
2. *$(n, q) = (4, 2)$ and S is conjugate to $\mathrm{GU}_1(q) \wr \mathrm{Sym}(4) = \mathrm{MU}_4(2)$ (so $S \cdot \mathrm{SU}_n(q) = \mathrm{GU}_n(q)$).*

Proof. The base sizes for (1) and (2) are verified by computation. Let us fix q and consider a minimal counterexample (n, S) to the statement of the theorem. So n is the smallest integer such that $\mathrm{GU}_n(q)$ is not solvable and $\mathrm{GU}_n(q)$ has an irreducible maximal solvable subgroup S with $b_S(S\cdot \mathrm{SU}_n(q)) > 3$ that is not as in (1) and (2) of the lemma. Since $\mathrm{PGU}_2(q) \cong \mathrm{PGL}_n(2)$, Lemma 2.13 and Theorem 3.12 allow us to assume $n \geq 3$. By Lemma 3.11, S is not quasi-primitive, so S has a normal subgroup L such that V is not homogeneous as $\mathbb{F}_q[L]$-module. Therefore, S and L satisfy the conditions of Lemma 2.6. So S stabilises a decomposition

$$V = V_1 \oplus \ldots \oplus V_k, \ k \geq 2 \tag{3.8}$$

such that (1) or (2) of Lemma 2.6 holds. Let us fix (3.8) to be such a decomposition with the largest possible k.

If (1) of Lemma 2.6 holds, then consider $S_1 := \mathrm{Stab}_S(V_1)|_{V_1} \leq \mathrm{GU}(V_1)$. By Clifford's Theorem S_1 acts irreducibly on V_1. Note that S_1 is quasi-primitive. Indeed, if S_1 is not quasi-primitive, then S_1 stabilises a decomposition

$$V_1 = V_{11} \oplus \ldots \oplus V_{1t}$$

for some $t \geq 2$ such that (1) or (2) of Lemma 2.6 holds. Therefore, since S is irreducible, it stabilises the decomposition

$$V = V_{11} \oplus \ldots \oplus V_{1t} \oplus \ldots \oplus V_{k1} \oplus \ldots \oplus V_{kt},$$

for which (1) or (2) of Lemma 2.6 holds contradicting the maximality of k in (3.8).

If $\dim V_1 = 1$, then S can be represented as a group of monomial matrices with respect to an orthonormal basis. By Lemma 3.14, this is possible if and only if $(n, q) = (4, 2)$ and $S = \mathrm{MU}_4(2)$. Assume $\dim V_1 = m \geq 2$. If $q \in \{2, 3, 5\}$ and S is conjugate to a subgroup of one of the groups listed in Lemma 3.20, then $b_S(S \cdot \mathrm{SU}_n(q)) \leq 3$ which is a contradiction. If S is not conjugate to a subgroup of a group from Lemma 3.20, then $\mathrm{GU}_m(q)$ is not solvable. Therefore, $S_1 \leq \mathrm{GU}(V_1)$ is not as S in (1) and (2) of the theorem, S_1 satisfies the condition of the theorem and $b_{S_1}(S_1 \cdot \mathrm{SU}(V_1)) \leq 3$ since (n, S) is a minimal counterexample. Thus, there exist $a, b \in \mathrm{SU}(V_1)$ such that $S_1 \cap S_1^a \cap S_1^b \leq Z(\mathrm{GU}(V_1))$. Applying Theorem 3.16 and Lemma 3.14 we obtain $b_S(S \cdot \mathrm{SU}_n(q)) \leq 3$.

Finally, let us assume part (2) of Lemma 2.6 holds. Let $U_i = V_{2i-1} \oplus V_{2i}$, so

$$V = U_1 \perp \ldots \perp U_{k/2}$$

and S transitively permutes the U_i. Indeed, since S acts on V by isometries, $(V_{2i-1})g$ and $(V_{2i})g$ cannot be mutually orthogonal for $g \in S$, which is possible if and only if $(V_{2i-1})g$ and $(V_{2i})g$ lie in the same U_j for some $j = 1, \ldots, k/2$. Transitivity follows from the irreducibility of S. Consider $S_1 := \mathrm{Stab}_S(U_1)|_{U_1}$. Notice that $\dim U_1 = 2m \geq 2$.

If $(2m, q) \in \{(2, 2), (2, 3)\}$, then, since S is a maximal solvable subgroup of $\mathrm{GU}(V)$, it must be conjugate to $\mathrm{GU}_{2m}(q) \wr \Gamma$ with $\Gamma \leq \mathrm{Sym}(k/2)$. In this case the theorem follows by Lemma 3.20. Otherwise $\mathrm{GU}(U_1)$ is not solvable. If $k > 8$, then $S_1 \leq \mathrm{GU}(U_1)$ satisfies the condition of the theorem and $b_{S_1}(\mathrm{GU}(U_1)) \leq 3$ since (n, S) is a minimal counterexample. Notice that S_1 is not as S in (1) of the lemma since S_1 is imprimitive (it is easy to check computationally that S in (1) of the lemma is primitive), and S_1 is not as S in (2) of the lemma since otherwise k is not maximal possible. Thus, there exist $a, b \in \mathrm{SU}(U_1)$ such that $S_1 \cap S_1^a \cap S_1^b \leq Z(\mathrm{GU}(U_1))$. Applying Theorem 3.16 and Lemma 3.14 we obtain $b_S(S \cdot \mathrm{SU}_n(q)) \leq 3$. Let $k \leq 8$. Consider $P := \mathrm{Stab}_S(V_1)|_{V_1} \leq \mathrm{GL}(V_1)$, so P is an irreducible solvable subgroup of $\mathrm{GL}(V_1)$ and there exist

$a, b \in \mathrm{SL}(V_1)$ such that $P \cap P^a \cap P^b \leq Z(\mathrm{GL}(V_1))$ by Theorem 2.11. Applying Lemma 3.19 we obtain $b_S(S \cdot \mathrm{SU}(V)) \leq 3$, which contradicts the assumption. □

Theorem 3.22. *Let $n \geq 2$ and $(n, q) \notin \{(2, 2), (2, 3)\}$. If $S \leq \mathrm{GSp}_n(q)$ is an irreducible maximal solvable subgroup, then one of the following holds:*

1. $b_S(S \cdot \mathrm{Sp}_n(q)) \leq 3$;
2. *$n = 2$, $q = 5$, S is an absolutely irreducible subgroup such that $S/Z(\mathrm{GSp}_2(q))$ is isomorphic to $2^2.\mathrm{Sp}_2(2)$, and $b_S(S \cdot \mathrm{Sp}_2(q)) = 4$;*
3. *$n = 4$, $q \in \{2, 3\}$, and S is the stabiliser of decomposition $V = V_1 \perp V_2$ with V_i non-degenerate and $b_S(S \cdot \mathrm{Sp}_n(q)) = 4$.*

Proof. If $n = 2$, then $\mathrm{GSp}_2(q) = \mathrm{GL}_2(q)$ and the theorem follows from Theorem 2.11. The following is verified by computation: if $n = 4$ and $q \in \{2, 3, 5\}$, then either $b_S(S \cdot \mathrm{Sp}_n(q)) \leq 3$ or S is as in (3).

Assume that n is minimal such that there exists a counterexample to the theorem: namely, (S, n, q) is such that $b_S(\mathrm{Sp}_n(q)) > 3$ and neither of (2)–(3) hold. If S is quasi-primitive, then it is not a counterexample by Theorem 3.12. Hence S is not quasi-primitive, so S has a normal subgroup L such that V is not $\mathbb{F}_q[L]$-homogeneous by Lemma 2.6. Therefore, S stabilises a decomposition

$$V = V_1 \oplus \ldots \oplus V_k \tag{3.9}$$

such that $\dim V_i = m$ for $i \in \{1, \ldots, k\}$, $k > 1$ and one of the following holds:

Case 1. $V = V_1 \perp \ldots \perp V_k$ with V_i non-degenerate for $i = 1, \ldots, k$. We do not require V_i to be the $\mathbb{F}_q[L]$-homogeneous components. In particular, this includes the case when (2) of Lemma 2.6 holds with $k > 2$.

Case 2. $V = V_1 \oplus V_2$ with V_i totally isotropic.

Case 1. Let us fix (3.9) to be a S-invariant orthogonal decomposition of V with the largest possible k. Let H_i be $\mathrm{Stab}_S(V_i)|_{V_i} \leq \mathrm{GSp}_m(q)$. Since S is irreducible, the H_i are pairwise conjugate in $\mathrm{GSp}_m(q)$, so, in suitable basis, $S \leq H_1 \wr \mathrm{Sym}(k)$. In other words, we could assume $H_i = H_1$ for all $i \in \{1, \ldots, k\}$. We keep the distinction between the H_i since we refer to this proof in Lemma 5.1 where the H_i are conjugate in $\Gamma\mathrm{Sp}_m(q)$ but not necessarily in $\mathrm{GSp}_m(q)$. This case splits into two subcases: $k \geq 3$ and $k = 2$.

Case 1.1: $k \geq 3$. Notice that $H := H_1$ is an irreducible maximal solvable subgroup of $\mathrm{GSp}_m(q)$. Moreover, H is not as S in (3) since otherwise $V_i = V_{i1} \perp V_{i2}$ with non-degenerate V_{ij} and S stabilises the decomposition

$$V = (V_{11} \perp V_{12}) \perp \ldots \perp (V_{k1} \perp V_{k2})$$

which contradicts maximality of k. Since $m < n$ and H is not a counterexample, we can assume that either there exist $x_1, x_2 \in \mathrm{Sp}_m(q)$ such that $H \cap H^{x_1} \cap H^{x_2} \leq Z(\mathrm{GSp}_m(q))$ or $H = \mathrm{GSp}_2(q)$ with $q \in \{2, 3, 5\}$. In the latter case we take $x_1 = x_2 = I_2$.

Let $\{v_1, u_1, \ldots, v_k, u_k\}$ be a basis of a $2k$-dimensional vector space over $\mathbb{F}_q$. Let y_1 and z_1 be the matrices of the linear transformations of this space defined by the formulae:

$$\begin{aligned}
(v_i)y_1 &= v_i - v_{i+1} \text{ for } i \in \{1, \ldots, k-1\};\\
(v_k)y_1 &= v_k;\\
(u_i)y_1 &= \sum_{j=1}^{i} u_i \text{ for } i \in \{1, \ldots, k\};
\end{aligned}$$

and

$$(v_1)z_1 = v_1; \qquad (u_1)z_1 = u_1 + v_2;$$

$$(v_i)z_1 = v_i - v_{i+1} \text{ for } i \in \{1, \dots, k-2\}; \qquad (u_i)z_1 = \sum_{j=1}^{i} u_j \text{ for } i \in \{2, \dots, k-1\};$$

$$(v_{k-1})z_1 = v_{k-1} + u_k; \qquad (u_k)z_1 = u_k;$$

$$(v_k)z_1 = v_1 + v_k + \sum_{j=2}^{k-1} u_j.$$

For example, if $k = 4$, then

$$y_1 = \left(\begin{array}{cc|cc|cc|cc} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right), \quad z_1 = \left(\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Let β_i be a basis of V_i of shape (2.4) for $i \in \{1, \dots, k\}$ and let β be $\beta_1 \cup \ldots \cup \beta_k$. Let $y = I_{m/2} \otimes y_1$ and $z = I_{m/2} \otimes z_1$. It is routine to check that $y, z \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$.

Let $W_i = V_i(x_1 \otimes I_k)y$ for $i \in \{1, \dots, k\}$. Consider $g \in S \cap S^{(x_1 \otimes I_k)y}$, so g stabilises decompositions $V = V_1 \perp \ldots \perp V_k$ and $V = W_1 \perp \ldots \perp W_k$. Notice that W_i has non-zero projections on exactly $i+1$ of the V_j for $i \in \{1, \dots, k-2\}$. So g cannot map such W_i to others and, therefore, g stabilises subspaces $W_1, \dots, W_{k-2}$ and $\{W_{k-1}, W_k\}$. Thus, g stabilises $\{V_1, V_2\}$ and $V_3, \dots, V_k$. Notice that $\dim(V_1 \cap W_1) = m/2$ and $\dim(V_2 \cap W_1) = 0$. So $(V_1)g = V_1$ since $(V_1 \cap W_1)g = (V_1)g \cap (W_1)g = (V_1)g \cap W_1 \neq \{0\}$. The same argument for $V_n \cap W_n$ shows that $(W_n)g = W_n$. Hence g stabilises all V_i and W_i for $i \in \{1, \dots, k\}$; in particular

$$g = \mathrm{diag}[g_1, \dots, g_k]$$

for $g_i \in H_i$.

Now let us show that $g_i \in H_i \cap H_i^{x_1}$ for all $i \in \{1, \dots, k\}$. Since $g \in S \cap S^{(x_1 \otimes I_k)y}$, we obtain $g = h^{(x_1 \otimes I_k)y}$, where $h \in S \cap S^{((x_1 \otimes I_k)y)^{-1}}$. The same arguments as above show that $h = \mathrm{diag}[h_1, \dots, h_k]$ with $h_i \in H_i$. Denote $h_i^{x_1}$ by $\hat{h}_i$ and $h^{x_1 \otimes I_k} = \mathrm{diag}[\hat{h}_1, \dots, \hat{h}_k]$ by $\hat{h}$, so $g = \hat{h}^y$. Let $\hat{h}_i = \begin{pmatrix} h_{(i,1)} & h_{(i,2)} \\ h_{(i,3)} & h_{(i,4)} \end{pmatrix}$, where $h_{(i,j)} \in \mathrm{GL}_{m/2}(q)$.

Consider the last $(m \times m)$-row of $g = \hat{h}^y$. Calculations show that it is

$$(0, \dots, 0, g_k) = (A_1, A_2, \dots, A_k)$$

with

$$A_i = \begin{pmatrix} 0 & h_{(k,2)} \\ 0 & h_{(k,4)} - h_{(k-1,4)} \end{pmatrix} \qquad \text{for } i \in \{1, \dots, k-2\},$$

$$A_{k-1} = \begin{pmatrix} 0 & h_{(k,2)} \\ -h_{(k-1,3)} & h_{(k,3)} - h_{(k-1,4)} \end{pmatrix},$$

$$A_k = \begin{pmatrix} h_{(k,1)} & h_{(k,2)} \\ h_{(k,3)} - h_{(k-1,3)} & h_{(k,4)} \end{pmatrix}.$$

So, $h_{(k-1,4)} = h_{(k,4)}$; $h_{(k,2)} = h_{(k-1,3)} = 0$ and $\hat{h}_k = g_k$. Consider the $(k-1)$-th $(m \times m)$-row of $g = \hat{h}^y$. As above, we obtain

$$\begin{aligned} h_{(k-2,4)} &= h_{(k-1,4)};\\ h_{(k-2,1)} &= h_{(k-1,1)};\\ h_{(k-2,3)} &= h_{(k-1,2)} = 0 \end{aligned}$$

and $\hat{h}_{k-1} = g_{k-1}$.

Continuing in the same way we obtain for all $i, j \in \{1, \dots, k\}$:

$$\begin{aligned} \hat{h}_i &= g_i;\\ h_{(i,1)} &= h_{(j,1)};\\ h_{(i,4)} &= h_{(j,4)}. \end{aligned} \tag{3.10}$$

Also

$$\hat{h}_1 = \begin{pmatrix} h_{(1,1)} & h_{(1,2)} \\ 0 & h_{(1,4)} \end{pmatrix};\ \hat{h}_k = \begin{pmatrix} h_{(k,1)} & 0 \\ h_{(k,3)} & h_{(k,4)} \end{pmatrix}$$

and

$$\hat{h}_i = \begin{pmatrix} h_{(i,1)} & 0 \\ 0 & h_{(i,4)} \end{pmatrix} \text{ for } 1 < i < k.$$

Hence $g_i \in H_i \cap H_i^{x_1}$.

Assume now that $g \in S \cap S^{(x_2 \otimes I_k)z}$, $g = t^{(x_2 \otimes I_k)z}$, where $t \in S \cap S^{((x_2 \otimes I_k)z)^{-1}}$. Using similar arguments to above, we obtain

$$g = \mathrm{diag}[g_1, \dots, g_k] = \mathrm{diag}[\hat{t}_1, \dots, \hat{t}_k],$$

where $\hat{t}_i \in H_i^{x_2}$ is defined analogously to $\hat{h}_i \in H_i^{x_1}$. In addition,

$$t_{(k-1,4)} = t_{(k,1)} \text{ and } t_{(k,3)} = t_{(1,2)} = 0. \tag{3.11}$$

Therefore, if $g \in S \cap S^{(x_1 \otimes I_k)y} \cap S^{(x_2 \otimes I_k)z}$, then, by (3.10) and (3.11), $g = \mathrm{diag}[g_1, \dots, g_k]$ and $g_i = g_1 \mathrm{diag}[\delta, \delta] \in H \cap H^{x_1} \cap H^{x_2}$ with $\delta \in \mathrm{GL}_{m/2}(q)$ for all $i \in \{1, \dots, k\}$. If $m = 2$, then g is scalar, so $S \cap S^{(x_1 \otimes I_k)y} \cap S^{(x_2 \otimes I_k)z} \leq Z(\mathrm{GSp}_n(q))$. If $m > 2$, then H is not a counterexample, so $g_i \in H \cap H^{x_1} \cap H^{x_2} \leq Z(\mathrm{GSp}_m(q))$ and $g \in Z(\mathrm{GSp}_n(q, \mathbf{f}_\beta))$.

Case 1.2: $k = 2$. Let $H = H_1$. Thus, either H is not a counterexample, so there exist $x_1, x_2 \in \mathrm{Sp}_m(q)$ such that $H \cap H^{x_1} \cap H^{x_2} \leq Z(\mathrm{GSp}_m(q))$, or $H \leq \mathrm{GSp}_2(q)$ with $q \in \{2, 3, 5\}$. The latter was discussed at the beginning of the proof. Assume the former holds. Let

$$y = I_{m/2} \otimes \left(\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right),\ z = I_{m/2} \otimes \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

It is routine to check that $y, z \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$. Denote $(V_i)y$ by W_i and $(V_i)z$ by U_i for $i = 1, 2$. We claim that if $g \in S \cap S^{(x_1 \otimes I_2)y} \cap S^{(x_2 \otimes I_2)z}$, then g stabilises V_i, $i = 1, 2$. Assume the opposite, so $(V_1)g = (V_2)$. Therefore, $(W_1)g = W_2$ and $(U_1)g = U_2$. Thus,

$$(V_1 \cap W_1)g = (V_1)g \cap (W_1)g = (V_2 \cap W_2)$$

and

$$(V_1 \cap U_1)g = (V_1)g \cap (U_1)g = (V_2 \cap U_2).$$

Notice that $(V_1 \cap W_1) = (V_1 \cap U_1)$ but $(V_2 \cap W_2) \neq (V_2 \cap U_2)$ which is a contradiction. Therefore, $g = \mathrm{diag}[g_1, g_2]$ where $g_i \in H_i$. Also, $g = h^y$ where $h \in S^{y^{-1}} \cap S^{(x_1 \otimes I_2)}$ and $g = t^z$ where $t \in S^{z^{-1}} \cap S^{(x_2 \otimes I_2)}$. It is routine to check, using arguments as above, that $h = \mathrm{diag}[h_1, h_2]$ with $h_i \in H_i^{x_1}$ and $t = \mathrm{diag}[t_1, t_2]$ with $t_i \in H_i^{x_2}$.

Now calculations as in **Case (1.1)** show that

$$g = \left(\begin{array}{cc|cc} h_{(1,1)} & h_{(1,2)} & 0 & 0 \\ 0 & h_{(1,4)} & 0 & 0 \\ \hline 0 & 0 & h_{(2,1)} & 0 \\ 0 & 0 & h_{(2,3)} & h_{(2,4)} \end{array}\right) = \left(\begin{array}{cc|cc} t_{(1,1)} & t_{(1,2)} & 0 & 0 \\ 0 & t_{(1,4)} & 0 & 0 \\ \hline 0 & 0 & t_{(2,1)} & t_{(2,2)} \\ 0 & 0 & 0 & t_{(2,4)} \end{array}\right)$$

for some $h_{(i,j)}, t_{(i,j)} \in \mathrm{GL}_{m/2}(q)$ with

$$h_{(1,1)} = h_{(2,1)}; \qquad t_{(1,1)} = t_{(2,4)};$$
$$h_{(1,4)} = h_{(2,4)}; \qquad t_{(1,4)} = t_{(2,1)}.$$

So

$$g = \left(\begin{array}{cc|cc} g_{(1,1)} & g_{(1,2)} & 0 & 0 \\ 0 & g_{(1,1)} & 0 & 0 \\ \hline 0 & 0 & g_{(1,1)} & 0 \\ 0 & 0 & 0 & g_{(1,1)} \end{array}\right); \ g_1 = \begin{pmatrix} g_{(1,1)} & g_{(1,2)} \\ 0 & g_{(1,1)} \end{pmatrix}; \ g_2 = \begin{pmatrix} g_{(1,1)} & 0 \\ 0 & g_{(1,1)} \end{pmatrix}$$

with $g_i \in H_i \cap H_i^{x_1} \cap H_i^{x_2}$. Since $H_1 \cap H_1^{x_1} \cap H_1^{x_2} \leq Z(\mathrm{GSp}_m(q))$, we obtain $g_1 = g_2 \in Z(\mathrm{GSp}_m(q))$ and $g \in Z(\mathrm{GSp}_n(q))$.

Case 2. Let H_i be $\mathrm{Stab}_S(V_i)|_{V_i} \leq \mathrm{GL}_m(q)$. Thus, by Theorem 2.11, either $H_i \leq \mathrm{GL}_2(q)$ with $q \in \{2, 3, 5\}$ or there exist $x_1, x_2 \in \mathrm{GL}_m(q)$ such that $H_1 \cap H_1^{x_1} \cap H_1^{x_2} \leq Z(\mathrm{GL}_m(q))$. In the first case the theorem is verified by computation, so assume that the second case holds.

Fix β to be a basis of V as in (2.4) with $\langle f_1, \ldots, f_m \rangle = V_1$ and $\langle e_1, \ldots, e_m \rangle = V_2$. Let y be $I_m \otimes \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, and let X_i be $\mathrm{diag}[x_i, (x_i^{-1})^\top]$ for $i = 1, 2$. Notice that $y, X_1, X_2 \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$. Consider $g \in S \cap S^{X_1 y} \cap S^{X_2}$. By Lemma 2.16, $g = \mathrm{diag}[g_1, g_1]$ with $g_1 \in H_1 \cap H_1^{x_1} \cap H_1^{x_2}$, so $g \in Z(\mathrm{GL}_n(q)) \cap \mathrm{GSp}_n(q) = Z(\mathrm{GSp}_n(q))$. □

4. General case: unitary groups

As we mentioned in the introduction, it so more convenient to work with $\Gamma\mathrm{U}_n(q)$ instead of $\mathrm{P}\Gamma\mathrm{U}_n(q)$ to prove Theorem B1. So, in this section S is a maximal solvable subgroup of $\Gamma\mathrm{U}_n(q) = \mathrm{GU}_n(q) \rtimes \langle \phi_\beta \rangle$ where β is an orthonormal basis of $(V, \mathbf{f})$. Our goal is to prove Theorem B1 which we reformulate in the following equivalent way.

Theorem B1. *Let $n \geq 3$ and (n, q) is not $(3, 2)$. If S is a maximal solvable subgroup of $\Gamma\mathrm{U}_n(q)$, then one of the following holds:*

(1) $b_S(S \cdot \mathrm{SU}_n(q)) \leq 4$, *so* $\mathrm{Reg}_S(S \cdot \mathrm{SU}_n(q), 5) \geq 5$*;*
(2) $(n, q) = (5, 2)$ *and S is the stabiliser in $\Gamma\mathrm{U}_n(q)$ of a totally isotropic 1-dimensional subspace of the natural module,* $b_S(S \cdot \mathrm{SU}_n(q)) = 5$ *and* $\mathrm{Reg}_S(S \cdot \mathrm{SU}_n(q), 5) \geq 5$.

Recall that $g^\dagger = (\overline{g}^\top)^{-1}$ for $g \in \mathrm{GL}_n(q^{\mathbf{u}})$, see the discussion after Definition 2.3 for details. To prove Theorem B1, we need the following lemma.

Lemma 4.1. *Let $(n, q, \mathbf{u})$ be such that $\mathrm{GL}_n(q^{\mathbf{u}})$ is not solvable and $(n, q, \mathbf{u}) \neq (2, 5, 1)$. If S is an irreducible maximal solvable subgroup of $\mathrm{GL}_n(q^{\mathbf{u}})$, then there exist $x, y \in \mathrm{SL}_n(q^{\mathbf{u}})$ such that*

$$S \cap S^x \cap (S^\dagger)^y \leq Z(\mathrm{GL}_n(q^{\mathbf{u}})).$$

Proof. If $b_S(S \cdot \mathrm{SL}_n(q^{\mathbf{u}})) = 2$, then there exists $x \in \mathrm{SL}_n(q^{\mathbf{u}})$ such that

$$S \cap S^x \leq Z(\mathrm{GL}_n(q^{\mathbf{u}})),$$

so y can be arbitrary. Therefore, it suffices to consider cases (1)–(6) from Theorem 2.11 only. In cases (1), (2) and (5), S is the normaliser of a Singer cycle, so $S \cdot \mathrm{SL}_n(q^{\mathbf{u}}) = \mathrm{GL}_n(q^{\mathbf{u}})$. Since all Singer cycles are conjugate in $\mathrm{GL}_n(q^{\mathbf{u}})$, $S^\dagger = S^g$ for some $g \in \mathrm{GL}_n(q^{\mathbf{u}})$, so the statement follows by Theorem 2.11. In cases (3) and (6) we have $S^\dagger = S$, so the statement follows since $b_S(S \cdot \mathrm{SL}_n(q)) \leq 3$ by Theorem 2.11. In case (4) the statement is verified by computation. □

Lemma 4.2. *Theorem* B1 *holds for* $n = 3$.

Proof. If S stabilises no non-zero proper subspace of V, then the statement follows by [12, Theorem 1.1].

Assume that S stabilises $U < V$ and S stabilises no non-zero proper subspace of U, so U is either totally isotropic or non-degenerate.

If U is totally isotropic, then $\dim U = 1$ since a maximal totally isotropic subspace of a non-degenerate unitary space of dimension n has dimension $[n/2]$. By Lemma 2.8, there exists a basis $\beta = \{f, v, e\}$ such that $\mathbf{f}_\beta$ is the permutation matrix for the permutation $(1, 3)$ and all elements in S_β have shape $\phi^j g$ with

$$g = \begin{pmatrix} \alpha_1^\dagger & * & * \\ 0 & \alpha_2 & * \\ 0 & 0 & \alpha_1 \end{pmatrix}$$

where $j \in \{0, 1, \dots, 2f - 1\}$, $\alpha_i \in \mathbb{F}_{q^2}^*$ and $\alpha_2^{q+1} = 1$. Let η be a generator of $\mathbb{F}_{q^2}^*$. For q even let $\delta = 1$, for q odd let $\delta = \eta^{-(q+1)/2}$, so $\delta \cdot \delta^\dagger = \delta^{1-q} = -1$. The matrix $x = \mathrm{diag}(\delta^\dagger, 1, \delta)\mathbf{f}_\beta$ lies in $\mathrm{SU}_3(q, \mathbf{f}_\beta)$. It is routine to check that if $\varphi \in S_\beta \cap S_\beta^x$, then $\varphi = \phi^j g$ with $g = \mathrm{diag}(\alpha_1^\dagger, \alpha_2, \alpha_1)$. Let $\alpha \in \mathbb{F}_{q^2}$ be such that $\alpha + \alpha^q = 1$. It exists by Lemma 2.14. Let $\theta \in \mathbb{F}_{q^2}$ be η^{q-1} and let $y, z \in \mathrm{SU}_3(q, \mathbf{f}_\beta)$ be

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -\alpha & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ -\theta^{-1} & 1 & 0 \\ -\alpha & \theta & 1 \end{pmatrix}$$

respectively. If $\varphi \in S_\beta \cap S_\beta^x \cap S_\beta^y$, then φ stabilises $\langle e \rangle y = \langle e + v - \alpha f \rangle$, so $\alpha_1 = \alpha_2$ and $\varphi = \phi^j \alpha_1 I_3$. If $\varphi \in S_\beta \cap S_\beta^x \cap S_\beta^y \cap S_\beta^z$, then φ stabilises $\langle e \rangle z = \langle e + \theta v - \alpha f \rangle$, so $\theta^{p^j} \alpha_1 = \theta \alpha_1$. Thus, $\theta^{p^j - 1} = 1$ and $j = 0$ by Lemma 2.15, so $\varphi \in Z(\mathrm{GU}_3(q))$.

Assume U is non-degenerate, so S stabilises $U^\perp$ and we can assume that $\dim U = 1$. Let $\beta = \{f, e, v\}$, where $\{f, e\}$ is a basis of $U^\perp$ as in (2.3) and $U = \langle v \rangle$. Let $x, y, z \in \mathrm{SU}_3(q, \mathbf{f}_\beta)$ be

$$\begin{pmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\alpha & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ -\alpha & 1 & \theta^{-1} \\ -\theta & 0 & 1 \end{pmatrix}$$

respectively. If $\varphi \in S_\beta \cap S_\beta^x \cap S_\beta^y \cap S_\beta^z$, then φ stabilises $\langle v \rangle$, $\langle v - f \rangle$, $\langle v - e \rangle$ and $\langle v - \theta f \rangle$. Arguments as in the previous case show that $\varphi \in Z(\mathrm{GU}_3(q))$. □

Recall that $q = p^f$ where p is a prime and f is a positive integer.

Lemma 4.3. *Let (n, q) be such that $\mathrm{GU}_n(q)$ is not solvable. Let S be a maximal solvable subgroup of $\Gamma\mathrm{U}_n(q)$ and let $M = S \cap \mathrm{GU}_n(q)$. If S stabilises no non-zero proper subspace of V, then either M lie in an irreducible solvable subgroup of $\mathrm{GU}_n(q)$ or there exist $y, z \in \mathrm{SU}_n(q)$ such that $M \cap M^y \cap M^z \leq Z(\mathrm{GU}_n(q))$.*

Proof. If $M \leq \mathrm{GU}_n(q)$ is irreducible, then such y, z exist by Theorem 3.21. Assume that M is reducible. Hence there exists $0 < U_1 < V$ of dimension m such that $(U_1)M = U_1$ and U_1 is $\mathbb{F}_{q^2}[M]$-irreducible. Let $\varphi \in S$ be such that $M\varphi$ is a generator of S/M, so $|M\varphi| = |S : M| = r$. Let $U_i = U_1\varphi^{i-1}$ for $i \in \{1, \dots, r\}$. By Lemma 2.5, M is completely reducible stabilising each subspace of the decomposition

$$V = U_1 \oplus \ldots \oplus U_k;\ k = n/m.$$

In particular, φ permutes the U_i cyclically and k divides $|S : M|$.

If $m = 1$, then M is abelian and the lemma follows by Theorem 2.10. So further we assume $m \geq 2$. Let us fix q and consider a minimal counterexample (n, S). So n is the smallest integer such that $\mathrm{GU}_n(q)$ is not solvable and $\Gamma\mathrm{U}_n(q)$ has a maximal solvable subgroup S stabilising no non-zero proper subspaces of V, $M = S \cap \mathrm{GU}_n(q)$ is reducible and $b_M(M \cdot \mathrm{SU}_n(q)) > 3$.

If V is $\mathbb{F}_{q^2}[M]$-homogeneous, then M stabilises each of the V_i in a decomposition $V = V_1 \oplus \ldots \oplus V_k$ as in Lemma 2.6 and all the V_i are irreducible $\mathbb{F}_{q^2}[M]$-submodules of V by [1, (5.2) and (5.3)]. Let M_i be the restriction of M on V_i.

First, assume that the V_i are non-degenerate, so each M_i is a solvable irreducible subgroup of $\mathrm{GU}_m(q)$. Let β_i be an orthonormal basis of V_i and $\beta = \cup_{i=1}^k \beta_i$. If $\mathrm{GU}_m(q)$ is solvable, then M lies in $\mathrm{GU}_m(q) \wr \mathrm{Sym}(k)$ and the lemma follows by Lemma 3.20. If M_1 is a subgroup of S as in (1) or (2) of Theorem 3.21, then $k = 2$ since $k \leq |S : M|$ and q is a prime. So (n, q) is either $(4, 5)$ or $(8, 2)$. In this case, the lemma is verified using computations. Otherwise, there exist $y_1, z_1 \in \mathrm{SU}(V_1)$ such that $M_1 \cap M_1^{y_1} \cap M_1^{z_1} \leq Z(\mathrm{GU}(V_1))$. Let $y = \mathrm{diag}[y_1, I_m, \dots, I_m]$ and $z = \mathrm{diag}[z_1, I_m, \dots, I_m]$ with respect to β, so $y, z \in \mathrm{SU}_n(q)$. Then the restriction of $M \cap M^y \cap M^z$ on V_1 is $M_1 \cap M_1^{y_1} \cap M_1^{z_1} \leq Z(\mathrm{GU}(V_1))$. So $M \cap M^y \cap M^z \leq Z(\mathrm{GU}_n(q))$ since M is homogeneous.

Now let the V_i be totally isotropic, so each M_i is a solvable irreducible subgroup of $\mathrm{GL}_m(q^2)$. Hence there exist $y_1, z_1 \in \mathrm{SL}(V_1)$ such that $M_1 \cap M_1^{y_1} \cap M_1^{z_1} \leq Z(\mathrm{GL}(V_1))$ by Theorem 2.11. Let $\{f_1, \dots, f_m\}$ and $\{e_1, \dots, e_m\}$ be the bases of V_1 and V_2 respectively such that $\beta_1 = \{f_1, \dots, f_m, e_1, \dots, e_m\}$ is a basis of $V_1 \oplus V_2$ as in (2.3). Hence, if g is an element of the restriction of M on $V_1 \oplus V_2$, then $g = \mathrm{diag}[g_1, g_1^{\dagger}]$. Let β_2 be a basis of $(V_1 \oplus V_2)^{\perp}$ and let $\beta = \beta_1 \cup \beta_2$. Let $y = [y_1, y_1^{\dagger}, I_{n-2m}]$ and $z = [z_1, z_1^{\dagger}, I_{n-2m}]$, so $y, z \in \mathrm{SU}_n(q, \mathbf{f}_\beta)$. Then the restriction of $M \cap M^y \cap M^z$ on V_1 is $M_1 \cap M_1^{y_1} \cap M_1^{z_1} \leq Z(\mathrm{GL}(V_1))$. So $M \cap M^y \cap M^z \leq Z(\mathrm{GU}_n(q))$ since M is homogeneous.

Now assume that V is not $\mathbb{F}_{q^2}[M]$-homogeneous, so, by Lemma 2.6, S stabilises a decomposition of $V = V_1 \oplus \ldots \oplus V_k$ as in Lemma 2.6 and all the V_i are $\mathbb{F}_{q^2}[M]$-submodules of V.

First, assume that $k = 2$ and the V_i are totally isotropic. Hence there exist $y_1, z_1 \in \mathrm{SL}(V_1)$ such that $M_1 \cap M_1^{y_1} \cap M_1^{z_1} \leq Z(\mathrm{GL}(V_1))$ by Theorem 2.11 and Lemma 2.12. Let $\{f_1, \dots, f_m\}$ and $\{e_1, \dots, e_m\}$ be the bases of V_1 and V_2 respectively such that $\beta = \{f_1, \dots, f_m, e_1, \dots, e_m\}$ is a basis of $V = V_1 \oplus V_2$ as in (2.3). Hence, if g is an element of M, then $g = \mathrm{diag}[g_1, g_1^{\dagger}]$ with respect to β. Now $y, z \in \mathrm{SU}_n(q, \mathbf{f}_\beta)$ such that $M \cap M^y \cap M^z \leq Z(\mathrm{GU}_n(q))$ exist by the proof of Lemma 3.19.

Now assume that either $k \geq 2$ and the V_i are non-degenerate or $k \geq 4$ and the V_i are totally isotropic. Hence S stabilises the decomposition (and M stabilises each of its summands)

$$V = W_1 \perp \ldots \perp W_t$$

where $t = k$ and $W_i = V_i$ if the V_i are non-degenerate and $t = k/2$ and $W_i = V_{2i-1} \oplus V_{2i}$ otherwise. Since S stabilises no non-zero proper subspaces of V, the stabiliser of W_i in S induces a subgroup $S_i \leq \Gamma\mathrm{U}_{n/t}(q)$ that stabilises no non-zero proper subspaces of W_i. Let M_i be $S_i \cap \mathrm{GU}_{n/t}(q)$. Since (n, S) is a minimal counterexample, either M_i lies in an irreducible solvable subgroup of $\mathrm{GU}_{n/t}(q)$ or there exist $y_i, z_i \in \mathrm{SU}_{n/t}(q)$ such that

$$M_i \cap M_i^{y_i} \cap M_i^{z_i} \leq Z(\mathrm{GU}_{n/t}(q)). \tag{4.1}$$

If $\mathrm{GU}_{n/t}(q)$ is solvable, then M lies in $\mathrm{GU}_m(q) \wr \mathrm{Sym}(k)$ and the lemma follows by Lemma 3.20. If M_1 is a subgroup of S as in (1) or (2) of Theorem 3.21, then $k = 2$ since $k \leq |S : M|$ and q is a prime. So (n, q) is either $(4, 5)$ or $(8, 2)$. In this case, the lemma is verified by computation. So, by Theorem 3.21, we can assume that there exist $y_i, z_i \in \mathrm{SU}_{n/t}(q)$ such that (4.1) holds.

Let D be the subgroup of all diagonal with respect to an orthonormal basis matrices in $\mathrm{GU}_t(q)$. As we noted before, if q is prime, then $k = 2$. Hence $\mathbf{F}(\mathrm{GU}_t(q)) = Z(\mathrm{GU}_t(q))$ unless $t = k = 2$ and $q \in \{2, 3\}$. If this is not a case, then, by Theorem 2.10, there exists $a \in \mathrm{SU}_t(q)$ such that $D \cap D^a \leq Z(\mathrm{GU}_t(q))$ and S is not a counterexample by Lemma 3.18.

Finally, assume that $t = k = 2$ and $q \in \{2, 3\}$, so $m > 2$ as otherwise $\mathrm{GU}_{n/t}(q)$ is solvable and the lemma follows by Lemma 3.20. Denote $y = \mathrm{diag}[y_1, y_2]$ and $z = \mathrm{diag}[z_1, z_2]$. Let $\beta_i = \{f_{i,1}, \ldots f_{[i,m/2]}, \underline{x_i}, e_{i,1}, \ldots, e_{i,[m/2]}\}$ for $i \in \{1, 2\}$ be a basis for V_i as in (2.3) and let $\beta = \beta_1 \cup \beta_2$. Here the underlined part is present if m is odd and absent otherwise. Let $a \in \mathrm{GL}_n(q^2)$ be such that

$$\begin{aligned} (f_{1,j})a &= f_{1,j} + f_{2,j} \text{ for } j \in \{1, \ldots, [m/2]\} \\ (e_{2,j})a &= -e_{1,j} + e_{2,j} \text{ for } j \in \{1, \ldots, [m/2]\} \end{aligned}$$

and a stabilises all remaining vectors in β. It is routine to check that $a \in \mathrm{SU}_n(q)$. We claim that $M \cap M^{ya} \cap M^z \leq Z(\mathrm{GU}_n(q))$. For simplicity, we consider the case when m is even; the case of odd m is fully analogous. Here

$$a_\beta = \begin{pmatrix} I_{m/2} & 0 & I_{m/2} & 0 \\ 0 & I_{m/2} & 0 & 0 \\ 0 & 0 & I_{m/2} & 0 \\ 0 & -I_{m/2} & 0 & I_{m/2} \end{pmatrix}.$$

Consider $h \in M \cap M^{ya}$. Hence $h = g^a$ for some $g \in M^y$. So g has shape

$$\begin{pmatrix} g_{11} & g_{12} & 0 & 0 \\ g_{13} & g_{14} & 0 & 0 \\ 0 & 0 & g_{21} & g_{22} \\ 0 & 0 & g_{23} & g_{24} \end{pmatrix}$$

with $g_{ij} \in M_n(q^2)$. Therefore,

$$h = g^a = \begin{pmatrix} g_{11} & g_{12} + g_{22} & g_{11} - g_{21} & -g_{22} \\ g_{13} & g_{14} & g_{13} & 0 \\ 0 & -g_{22} & g_{21} & g_{22} \\ g_{13} & g_{14} - g_{24} & g_{13} + g_{23} & g_{24} \end{pmatrix} \tag{4.2}$$

and, since $h \in M$, we obtain that $g_{13} = g_{22} = 0$, $g_{11} = g_{21}$ and $g_{14} = g_{24}$. In particular, it is easy to see now that $g^a = g \in M^y$. Assume, in addition, that $h \in M^z$, so $h = \mathrm{diag}[h_1, h_2]$ where $h_i \in M_i \cap M_i^{y_i} \cap M_i^{z_i} \leq Z(\mathrm{GU}_m(q))$. Note that $h_1 = h_2$ since h has shape (4.2) with $g_{11} = g_{21}$, so $h \in Z(\mathrm{GU}_n(q))$ and the claim follows. Hence S not a counterexample. □

Theorem 4.4. *Theorem* B1 *holds for* $n \geq 4$ *if* S *stabilises no non-zero proper subspace of* V.

Proof. The result follows by [12, Theorem 1.1] unless $n = 4$ and S lies in a maximal subgroup of type $\mathrm{Sp}_4(q)$ as in [12, Table 1]. We now consider this outstanding case.

Let $n = 4$ and $M = S \cap \mathrm{GU}_n(q)$. For $q \leq 4$ the theorem is verified by computation, so we assume $q \geq 5$. If S stabilises a decomposition of V as in Lemma 2.6, then the statement follows by [12, Table 2]. Hence we can assume that if $N \leq M$ is normal in S, then V is $\mathbb{F}_{q^2}[N]$-homogeneous. In particular, every characteristic abelian subgroup of M is cyclic by [33, Lemma 0.5].

Assume that M is reducible, so M stabilises non-zero $W < V$ such that W is $\mathbb{F}_{q^2}[M]$-irreducible and W is either non-degenerate or totally isotropic. If V is not $\mathbb{F}_{q^2}[M]$-homogeneous, then S stabilises a decomposition as in Lemma 2.6 which contradicts the assumption above, so V is $\mathbb{F}_{q^2}[M]$-homogeneous. Therefore, if $\dim W = 1$, then M is a group of scalars, so $S/Z(\mathrm{GU}_n(q))$ is cyclic and $b_S(S \cdot \mathrm{GU}_4(q)) \leq 2$ by Theorem 2.10. Hence we may assume that $\dim W = 2$ and W is either totally isotropic or non-degenerate.

First assume that $\dim W = 2$ and W is totally isotropic. By [1, (5.2)],

$$V = W_1 \oplus W_2$$

where W_i is a M-invariant submodule of V isometric to W, so we can assume $W_1 = W$. Let β be a basis as in (2.3) corresponding to this decomposition of V. Let $M_1 \leq \mathrm{GL}_2(q^2)$ be the restriction of M in W. By Theorem 2.11, since $q \geq 5$, either there exists $x_1 \in \mathrm{SL}_2(q^2)$ such that $M_1 \cap M_1^{x_1} \leq Z(\mathrm{GL}_2(q^2))$ or M_1 is a subgroup of the normaliser of a Singer cycle in $\mathrm{GL}_2(q^2)$. If x_1 as above exists, then $M \cap M^x \leq Z(\mathrm{GU}_n(q))$ where $x_\beta = \mathrm{diag}[x_1, x_1^\dagger]$, since V is $\mathbb{F}_q[M]$-homogeneous. Therefore, $b_S(S \cdot \mathrm{SU}_4(q)) \leq 4$ by Theorem 2.10.

Let M_1 be a subgroup of the normaliser of a Singer cycle in $\mathrm{GL}_2(q^2)$. Since $M \cong M_1$, it has a maximal abelian normal subgroup A of index at most 2, which is also characteristic. Hence V is $\mathbb{F}_{q^2}[A]$-homogeneous and the dimension of an irreducible $\mathbb{F}_{q^2}[A]$-submodule of V is odd by Lemma 3.6, so A is a group of scalars. So M is cyclic modulo scalars and we obtain $b_S(S \cdot \mathrm{SU}_4(q)) \leq 4$ by applying Theorem 2.10 twice.

Now let us assume that either $\dim W = 2$ and W is non-degenerate or M is irreducible (here we let $W = V$, so $\dim W = 4$). Let $m = \dim W$. Since every characteristic abelian subgroup of M is cyclic, M satisfies the conditions of [33, Corollary 1.4]. In particular, in the notation of Lemma 3.2, the following hold:

(1) $F = ET$, $Z = E \cap T$ and $T = C_F(E)$;
(2) a Sylow subgroup of E is either cyclic of prime order or extra-special;
(3) there exists $U \leq T$ of index at most 2 with U cyclic and characteristic in M, and $C_T(U) = U$;
(4) $EU = C_F(U)$ is characteristic in M.

Since U is characteristic in M, V is $\mathbb{F}_{q^2}[U]$-homogeneous, so, by Lemma 3.6, U is a group of scalars, $T = U$ and $M = C = C_M(U)$. Let e be such that $e^2 = |E/Z|$. Let $0 < L \leq W$ be an $\mathbb{F}_{q^2}[EU]$-submodule. By [33, Corollary 2.6],

$$m = e \cdot \dim L.$$

Thus, $e \in \{1, 2, 4\}$, so E is either cyclic or an extra-special 2-group. By the proof of (vii) and (ix) of [33, Corollary 1.10], $F = C_M(E/Z)$ and M/F is trivial for $e = 1$ and isomorphic to a subgroup of $\mathrm{Sp}_e(2)$ for $e \in \{2, 4\}$.

If $e = 1$, then $F = U$ is self-centralising (since the centraliser of the Fitting subgroup of a solvable group lies in the Fitting subgroup) and W is $\mathbb{F}_{q^2}[U]$-irreducible by [33, Lemma 2.2], which is a contradiction, since U is a group of scalars. Therefore, $e \in \{2, 4\}$.

If $e = 4$, then, by the proof of Lemma 3.10, $M = M_1 \cdot Z(\mathrm{GU}_4(q))$ and M_1 lies in the normaliser of a symplectic-type subgroup of $\mathrm{GU}_4(p^t)$ for some $t \leq f$. Hence $b_M(M \cdot \mathrm{SU}_4(q)) \leq 2$ for $q > 3$ by [12, Table 2] and $b_S(S \cdot \mathrm{SU}_4(q)) \leq 4$ by Theorem 2.10. For $q \leq 3$ the statement is verified by computation.

Let $e = 2$. Therefore, $|M| = |U| \cdot |E/Z| \cdot |M/F|$ divides

$$(q + 1) \cdot e^2 \cdot |\mathrm{Sp}_2(2)| = 24(q + 1).$$

So $|S|$ divides $24(q + 1) \cdot 2f$ and $|S/Z(\mathrm{GU}_4(q))|$ divides $48f$. We claim that

$$\hat{Q}((S \cdot \mathrm{SU}_4(q)/Z(\mathrm{GU}_4(q)), 4) < 1$$

where $\hat{Q}(G, c)$ is as in (2.10) and $H = S/Z(\mathrm{GU}_4(q))$. By Lemma 2.18, if $x_1, \ldots, x_k$ represent distinct G-classes such that $\sum_{i=1}^{k} |x_i^G \cap H| \leq A$ and $|x_i^G| \geq B$ for all $i \in \{1, \ldots, k\}$, then

$$\sum_{i=1}^{m} |x_i^G| \cdot \mathrm{fpr}(x_i)^c \leq B \cdot (A/B)^c.$$

We take $A = 48f \geq |H| \geq \sum_{i=1}^{k} |x_i^G \cap H|$. For elements in $\mathrm{PGU}_4(q)$ of prime order with $s = \nu(x) \in \{1, 2, 3\}$ we use (2.12) as a lower bound for $|x_i^G|$. If $x \in H \backslash \mathrm{PGU}_4(q)$ has prime order, then we use the corresponding bound for $|x^G|$ in [9, Corollary 3.49]. We take B to be the smallest of these bounds for $|x_i^G|$. For $q \geq 5$, such A and B are sufficient to obtain

$$\hat{Q}((S \cdot \mathrm{SU}_4(q)/Z(\mathrm{GU}_4(q)), 4) < 1,$$

so $b_S(S \cdot \mathrm{SU}_4(q)) \leq 4$. □

Theorem 4.5. *Theorem* B1 *holds for $n \geq 4$ if S stabilises a non-zero proper subspace of V.*

Proof. The proof proceeds in two steps. In **Step 1** we obtain three conjugates of S such that elements of their intersection have shape $\phi_\beta g$ for some basis β of V where $g \in \mathrm{GU}_n(q, \mathbf{f}_\beta)$ is diagonal or has few non-zero entries not on the diagonal. In **Step 2** we find a fourth conjugate of S such that the intersection of the four is a group of scalars.

Step 1. Fix a basis β of the unitary space $(V, \mathbf{f})$ as in Lemma 2.8, so $\mathbf{f}_\beta$ is as in (2.7) and elements of S take shape $\phi_\beta^j g$ with g as in (2.8) and $j \in \{0, 1, \ldots, 2f - 1\}$. We consider S as a subgroup of $\Gamma\mathrm{U}_n(q, \mathbf{f}_\beta)$. Let M be $S \cap \mathrm{GU}_n(q, \mathbf{f}_\beta)$. We obtain three conjugates of S such that their intersection consists of elements $\phi_\beta^j g$ where g is diagonal with respect to β.

Let γ_i be as in Lemma 2.8. Observe that $\mathbf{f}_\beta \mathbf{f}_\beta \overline{\mathbf{f}_\beta}^\top = \mathbf{f}_\beta$, so $\mathbf{f}_\beta \in \mathrm{GU}_n(q, \mathbf{f}_\beta)$. Notice that $\det(\mathbf{f}_\beta) = (-1)^{n_1 + \ldots + n_k}$. If $\sum_{i=1}^{k} n_i$ is odd, then one of the n_r is odd for some $r \in \{1, \ldots, k\}$. Let δ be as in the proof of Lemma 4.2, so $\delta\delta^\dagger = -1$. Notice that

$$h = \mathrm{diag}[I_{n_1}, \ldots, I_{n_{r-1}}, \delta^\dagger I_{n_r}, I_{n_r+1}, \ldots, I_{n_r+1}, \delta I_{n_r}, I_{n_{r-1}}, \ldots, I_{n_1}] \in \mathrm{GU}_n(q, \mathbf{f}_\beta)$$

has determinant $\det(\mathbf{f}_\beta)$. In particular, $x = h\mathbf{f}_\beta \in \mathrm{SU}_n(q)$. It is easy to see that if $g \in M$, so it has shape (2.8), then

$$g^x = \left(\begin{array}{ccc|ccc|ccc}
\gamma_1(g) & 0 & & & & & & & 0 \\
* & \ddots & & & & & & & \\
* & * & \gamma_k(g) & & & & & & \\ \cline{1-6}
* & \dots & * & \gamma_{k+1}(g) & 0 & & & & \\
* & \dots & * & & \ddots & & & & \\
* & \dots & * & 0 & & \gamma_{k+l}(g) & & & \\ \cline{4-9}
* & \dots & & & & * & \gamma_k(g)^\dagger & & 0 \\
* & \dots & & & & * & * & \ddots & \\
* & \dots & & & & * & * & * & \gamma_1(g)^\dagger
\end{array}\right).$$

Let $q \notin \{2, 3, 5\}$. By Lemmas 2.12, 4.1, 3.21 and 4.3 there exist $y_i, z_i \in \mathrm{SL}_{n_i}(q^2)$ for $i = 1, \dots, k$ and $y_i, z_i \in \mathrm{SU}_{n_i}(q)$ for $i = k+1, \dots, k+l$ such that

$$\gamma_i(M) \cap \gamma_i(M)^{y_i} \cap (\gamma_i(M)^\dagger)^{z_i} \leq Z(\mathrm{GL}_{n_i}(q^2)). \tag{4.3}$$

Notice that $\gamma_i(M)^\dagger = \gamma_i(M)$ for $i = k+1, \dots, k+l$. Denote by y and z the block-diagonal matrices

$$\begin{aligned}&\mathrm{diag}[y_1^\dagger, \dots, y_k^\dagger, y_{k+1}, \dots, y_{k+l}, y_k, \dots, y_1] \text{ and}\\ &\mathrm{diag}[z_1^\dagger, \dots, z_k^\dagger, z_{k+1}, \dots, z_{k+l}, z_k, \dots, z_1]\end{aligned} \tag{4.4}$$

respectively. It is routine to check that $y, z \in \mathrm{SU}_n(q, \mathbf{f}_\beta)$.

Therefore, if $g \in M \cap M^{xz}$, then g is the block-diagonal matrix

$$\mathrm{diag}[g_1^\dagger, \dots, g_k^\dagger, g_{k+1}, \dots, g_{k+l}, g_k, \dots, g_1], \tag{4.5}$$

where $g_i \in \gamma_i(M) \cap (\gamma_i(M)^\dagger)^{z_i}$ for $i = 1, \dots, k+l$. Thus, if $g \in M \cap M^y \cap M^{xz}$, then g has shape (4.5) where

$$g_i \in \gamma_i(M) \cap \gamma_i(M)^{y_i} \cap (\gamma_i(M)^\dagger)^{z_i} \leq Z(\mathrm{GL}_{n_i}(q^2)) \text{ for } i = 1, \dots, k+l.$$

So, by Lemma 2.9, we can assume that elements in $\gamma_i(S) \cap \gamma_i(S)^{y_i} \cap (\gamma_i(S)^\dagger)^{z_i}$ have shape $\phi^j g_i$ with $g_i \in Z(\mathrm{GL}_{n_i}(q^2))$. Thus, if $\varphi \in S \cap S^y \cap S^{xz}$, then $\varphi = \phi^j g$ with g as in (4.5) and $g_i \in Z(\mathrm{GL}_{n_i}(q^2))$. Denote $S \cap S^y \cap S^{xz}$ by $\tilde{S}$ and $M \cap \tilde{S}$ by $\tilde{M}$.

If $q \in \{2, 3, 5\}$, then it may be that $\gamma_{k+i}(M) \in \{\mathrm{GU}_2(q), \mathrm{GU}_3(2), \mathrm{MU}_4(2)\}$. Recall that $\mathrm{MU}_n(q)$ is defined in Lemma 3.14. In view of Theorem 3.21, and since $\mathrm{GU}_2(2)$, $\mathrm{GU}_2(3)$ and $\mathrm{GU}_3(2)$ are solvable, elements y_{k+i} and z_{k+i} as in (4.3) do not exist. If there is more than one such $\gamma_{k+i}(M)$, say

$$\gamma_{k+i_1}(M), \dots, \gamma_{k+i_\mu}(M),$$

then we join them in pairs, and there is one such unpaired group if μ is odd. Let $H_1, H_2 \in \{\mathrm{GU}_2(q), \mathrm{GU}_3(2), \mathrm{MU}_4(2)\}$ and let ν_j for $j \in \{1, 2\}$ be the corresponding degree of H_i, so $H_i \leq \mathrm{GU}_{\nu_j}(q)$. Let

$$H = H_1 \times H_2 = \{\mathrm{diag}[h_1, h_2] \mid h_j \in H_j\} \leq \mathrm{GU}_{\nu_1+\nu_2}(q).$$

Computations show that

$$b_H(H \cdot \mathrm{SU}_{\nu_1+\nu_2}(q)) \leq 3.$$

Therefore, we can assume that there is at most one such $\gamma_{k+i}(S)$, so $\mu \leq 1$. Denote the degree of such $\gamma_{k+i}(S)$ by ν, so $2 \leq \nu \leq 4$. Repeating the argument above for the rest of $\gamma_i(M)$ and $\gamma_i(S)$, we obtain that if $\varphi \in \tilde{S}$, then $\varphi = \phi^j g$ with g as in (4.5) and either all $g_i \in Z(\mathrm{GL}_{n_i}(q^2))$ (if $\mu = 0$) or all but one $g_i \in Z(\mathrm{GL}_{n_i}(q^2))$ and one g_i (for $i > k$) is a $(\nu \times \nu)$ matrix (if $\mu = 1$).

Remark 4.6. It may be that some $\gamma_{k+i}(M)$ have degree 1, so $\gamma_{k+i}(S) \leq \Gamma\mathrm{U}_1(q)$. We can treat them together. Indeed, assume that $\gamma_{k+i}(M)$ has degree 1 for $i = 1, \dots, \zeta \leq l$. Define ${\gamma_{k+1}}' : S \to \Gamma\mathrm{U}_\zeta(q)$ by

$$ {\gamma_{k+1}}'(\phi^j g) = \phi^j \mathrm{diag}(\gamma_1(g), \dots, \gamma_\zeta(g)) $$

for $g \in M$ and $j \in \{0, 1, \dots, 2f-1\}$. Hence the group $T = {\gamma_{k+1}}'(M)$, consisting of diagonal matrices, is an abelian subgroup of $T \cdot \mathrm{SU}_\zeta(q) = \mathrm{GU}_\zeta(q)$. If $q > 3$, then by Theorem 2.10 there exists $g \in \mathrm{SU}_\zeta(q)$ such that $T \cap T^g \leq \mathbf{F}(\mathrm{GU}_\zeta(q)) = Z(\mathrm{GU}_\zeta(q))$, where $\mathbf{F}(G)$ is the Fitting subgroup of a finite group G. So (4.3) holds for ${\gamma_{k+1}}'(M)$ and we can replace $\gamma_1, \dots, \gamma_\zeta$ with ${\gamma_{k+1}}'$ of degree ζ. Finally, suppose $q \leq 3$. Notice that

$$ T \rtimes \langle \phi \rangle < ((\mathrm{GU}_1(q))^{(1/2+(-1)^{\zeta-1}/2)} \times (\mathrm{GU}_2(q))^{[\zeta/2]}) \rtimes \langle \phi \rangle, $$

so if $\zeta > 1$, then S is not a maximal solvable subgroup of $\mathrm{GU}_n(q)$ since $\mathrm{GU}_2(q)$ is solvable. Therefore, we can assume that there is at most one $\gamma_{k+i}(S)$ of degree 1 in every case, so $\zeta \leq 1$.

We summarise the outcome of **Step 1**. Let μ be the number of $i \in \{1, \dots, l\}$ such that $\gamma_{k+i}(M)$ is a subgroup of one of the groups $\mathrm{GU}_2(q)$ for $q \in \{2, 3, 5\}$, $\mathrm{GU}_3(2)$ or $\mathrm{MU}_4(2)$. We may assume $\mu \in \{0, 1\}$. In particular, $\mu = 0$ if $q > 3$. There exist $x, y \in \mathrm{GU}_n(q)$ such that if $\varphi \in \tilde{S} = S \cap S^x \cap S^y$, then $\varphi = \phi^j g$ with g as in (4.5) and either all $g_i \in Z(\mathrm{GL}_{n_i}(q^2))$ (if $\mu = 0$) or all but one $g_i \in Z(\mathrm{GL}_{n_i}(q^2))$ and one g_i (for $i > k$) is a $(\nu \times \nu)$ matrix (if $\mu = 1$). Notice that ν is 2, 3, or 4 if the corresponding $\gamma_{k+i}(M)$ is $\mathrm{GU}_2(q)$, $\mathrm{GU}_3(2)$ and $\mathrm{MU}_4(2)$ respectively.

Step 2. We now find a fourth conjugate of S such that its intersection with $\tilde{S}$ lies in $Z(\mathrm{GU}_n(q))$. Let φ be an element of $\tilde{S}$.

Assume that S is such that $\mu = 0$. First we slightly modify the basis β from the first step. Recall that β is such that $\mathbf{f}_\beta$ is as in (2.7). Therefore,

$$ \begin{aligned} \beta = \{ & f_1^1, \dots, f_{n_1}^1, \dots, f_1^k, \dots, f_{n_k}^k, \\ & x_1^1, \dots, x_{n_{k+1}}^1, \dots, x_1^l, \dots, x_{n_{k+l}}^l, \\ & e_1^k, \dots, e_{n_k}^k, \dots, e_1^1, \dots, e_{n_1}^1 \}, \end{aligned} $$

where $(e_i^j, f_i^j) = 1$ and every other pair of vectors from β is mutually orthogonal. Let

$$ \begin{aligned} U_i &= \langle x_1^i, \dots, x_{n_{k+i}}^i \rangle, & i &= 1, \dots, l; \\ W_i &= \langle f_1^i, \dots, f_{n_i}^i, e_1^i, \dots, e_{n_i}^i \rangle, & i &= 1, \dots, k. \end{aligned} \tag{4.6} $$

Thus,

$$ V = (W_1 \perp \dots \perp W_k) \perp (U_1 \perp \dots \perp U_l), $$

where W_i, U_i are $\tilde{S}$-invariant subspaces and $\gamma_{k+i}(S) \leq \Gamma\mathrm{U}(U_i)$ for $i = 1, \dots, l$. By Lemma 2.1, we can choose for U_i the basis

$$ \beta_{1i} = \begin{cases} \{f_1^{k+i}, \dots, f_{m_i}^{k+i}, e_{m_i}^{k+i}, \dots, e_1^{k+i}\}, & \text{if } n_{k+i} = 2m_i; \\ \{f_1^{k+i}, \dots, f_{m_i}^{k+i}, x^{k+i}, e_{m_i}^{k+i}, \dots, e_1^{k+i}\}, & \text{if } n_{k+i} = 2m_i + 1. \end{cases} \tag{4.7} $$

By the first step

$$\gamma_{k+i}(\tilde{M}) \leq Z(\mathrm{GU}(U_i)),$$

so, by Lemmas 2.7 and 2.9, $\gamma_i(\varphi) = \phi^j_{\beta_{1i}} g_i$ with $g_i \in Z(\mathrm{GU}(U_i))$.

Now we renumber the basis vectors of the W_i from (4.6) and basis vectors of the U_i from (4.7) to obtain the basis

$$\beta_1 = \{f_1, \dots, f_m, x_1, \dots, x_t, e_m, \dots, e_1\},$$

where $m = \left(\sum_{i=1}^k n_k + \sum_{i=1}^l m_i\right)$ and t is the number of odd n_{k+i} for $i = 1, \dots, l$. In more detail, to obtain β_1 from β, we apply the following procedure:

- replace bases of U_i as in (4.6) by those as in (4.7), denote new basis by $\beta_{1/3}$;
- rearrange vectors as follows: first write down the f^i_j in the order they occur in $\beta_{1/3}$, then do the same with the x^i and then write the e^i_j in the order opposite to the f^i_j (so if f^i_j is the t-th entry of $\beta_{1/3}$, then e^i_j is the $(n-t+1)$-th entry of $\beta_{1/3}$). Denote new basis by $\beta_{2/3}$;
- relabel the f-vectors with just one index in the order they occur, do the same with the x-vectors and label the e-vectors such that $(f_i, e_i) = 1$.

We illustrate this procedure in the following example.

Example 4.7. Let $k = 2$, $l = 2$, $n_1 = 1$, $n_2 = 2$, $n_3 = 2$ and $n_4 = 3$. So

$$\begin{aligned} U_1 &= \langle x^1_1, x^1_2 \rangle = \langle f^3_1, e^3_1 \rangle \\ U_2 &= \langle x^2_1, x^2_2, x^2_3 \rangle = \langle f^4_1, x^4, e^4_1 \rangle \\ W_1 &= \langle f^1_1, e^1_1 \rangle \\ W_2 &= \langle f^2_1, f^2_2, e^2_1, e^2_2 \rangle \end{aligned}$$

and

$$\beta = \{f^1_1, f^2_1, f^2_2, x^1_1, x^1_2, x^2_1, x^2_2, x^2_3, e^2_1, e^2_2, e^1_1\}.$$

Hence

$$\beta_{1/3} = \{f^1_1, f^2_1, f^2_2, f^3_1, e^3_1, f^4_1, x^4, e^4_1, e^2_1, e^2_2, e^1_1\}$$

and

$$\beta_{2/3} = \{f^1_1, f^2_1, f^2_2, f^3_1, f^4_1, x^4, e^4_1, e^3_1, e^2_2, e^2_1, e^1_1\}.$$

The relabelling is

$$\begin{array}{cccccccccccccc} \beta_{2/3} & = \{ & f^1_1, & f^2_1, & f^2_2, & f^3_1, & f^4_1, & x^4, & e^4_1, & e^3_1, & e^2_2, & e^2_1, & e^1_1 & \} \\ \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \beta_1 & = \{ & f_1, & f_2, & f_3, & f_4, & f_5, & x_1, & e_5, & e_4, & e_3, & e_2, & e_1 & \}. \end{array} \tag{4.8}$$

We now resume the proof of Theorem 4.5. Notice that $\varphi \in \tilde{S}_{\beta_1}$ has shape $(\phi_{\beta_1})^j g$ with g as in (4.5) and $g_i \in Z(\mathrm{GL}_{n_i}(q^2))$. For simplicity we omit the subscripts and consider S and $\tilde{S}$ as subgroups in $\Gamma\mathrm{U}_n(q, \mathbf{f}_{\beta_1})$. Let $\phi^j g \in \tilde{S}$, so

$$g = \mathrm{diag}(\alpha^\dagger_1, \dots, \alpha^\dagger_m, \delta_1, \dots, \delta_t, \alpha_m, \dots, \alpha_1). \tag{4.9}$$

If

$$U_i = \begin{cases} \langle f_s, \dots, f_{s+m_i}, e_{s+m_i}, \dots, e_s \rangle, & \text{for } n_{k+i} = 2m_i, \\ \langle f_s, \dots, f_{s+m_i}, x_r, e_{s+m_i}, \dots, e_s \rangle, & \text{for } n_{k+i} = 2m_i + 1, \end{cases}$$

then $\alpha_s = \ldots = \alpha_{s+m_i} = \delta_r$ and $\alpha_s^{q+1} = 1$ since g is scalar on each U_i by the first step. If

$$W_i = \langle f_s, \ldots, f_{s+n_i}, e_{s+n_i}, \ldots, e_s \rangle, \tag{4.10}$$

then $\alpha_s = \ldots = \alpha_{s+n_i}$ since g is scalar on $\langle e_{s+n_i}, \ldots, e_s \rangle$ by the first step.

Remark 4.8. If $\alpha_i = \alpha_i^\dagger = \alpha_1$ for $i = 1, \ldots, m$, then g is not scalar if and only if $\zeta = 1$ in Remark 4.6. So, if there exists $\gamma_s(S)$ of degree 1, then we can assume, without loss of generality, that δ_1 is the corresponding entry (so $\gamma_s(S)$ acts on $\langle x_1 \rangle$). Therefore, if $\alpha_i = \alpha_i^\dagger = \delta_1$ for $i = 1, \ldots, m$, then $g = \delta_1 I_n \in Z(\mathrm{GU}_n(q, \mathbf{f}_{\beta_1}))$.

The remainder of our proof of **Step 2** splits into 3 cases:
Case 1. $\mu = 0$, $k > 0$;
Case 2. $\mu = 0$, $k = 0$;
Case 3. $\mu = 1$.
Each case splits into two or three subcases depending on other parameters. Our consideration of the subcases mostly follows the same pattern, so we omit details in some of them. Detailed proofs for each subcase can be found in [4]. In **Cases 1** and **2** we show $b_S(S \cdot \mathrm{SU}_n(q)) \leq 4$. In **Case 3** we show $b_S(S \cdot \mathrm{SU}_n(q)) \leq 4$ unless n is small ($q \in \{2, 3, 5\}$ here since $\mu = 1$). For small n the statement of Theorem B1 is verified by computation; we identify these values of n in **Case 3**.

Case 1. Let $\mu = 0$ and $k > 0$. So there is a totally singular S-invariant subspace

$$V_1 = \langle e_1, \ldots, e_{n_1} \rangle.$$

Recall that n_i is the degree of $\gamma_i(S)$ for $i \in \{1, \ldots, k + l\}$. Let $\alpha \in \mathbb{F}_{q^2}$ be such that $\alpha + \alpha^q = 1$, it exists by Lemma 2.14.

The three subcases we consider correspond to the following situations:
Case (1.1) $\dim W_i = 2$ for W_i in (4.6) and $i = 1, \ldots, k$;
Case (1.2) Condition of **Case (1.1)** does not hold and $l = 0$;
Case (1.3) Condition of **Case (1.1)** does not hold and $l > 0$.

Case (1.1). Assume that $\dim W_i = 2$ for W_i in (4.6) and $i = 1, \ldots, k$. Let η be a generator of $\mathbb{F}_{q^2}^*$ and let $\theta = \eta^{q-1}$. We redefine y from (4.4) to

$$\mathrm{diag}[A^\dagger, y_{k+1}, \ldots, y_{k+l}, A]$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ & & \ddots & & \\ 0 & \ldots & 0 & 1 & 0 \\ 1 & \ldots & \ldots & 1 & 1 \end{pmatrix}.$$

It is easy to see that $y \in \mathrm{SU}_n(q, \mathbf{f}_\beta)$. Let $x, z \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ be as in **Step 1**, so $\varphi \in \tilde{S}$ has shape $\phi^j g$ with g as in (4.9). Since S stabilises $\langle e_1 \rangle$, S^y stabilises $\langle e_1 \rangle y = \langle e_1 + \ldots + e_k \rangle$. Therefore,

$$((e_1)y)\varphi = (e_1 + \ldots + e_k)\phi^j g = \alpha_1 e_1 + \ldots + \alpha_k e_k = \lambda(e_1 + \ldots + e_k)$$

for some $\lambda \in \mathbb{F}_{q^2}^*$, so $\alpha_1 = \ldots = \alpha_k$.

Let $k \geq 2$. We claim that there exists $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ such that

$$\begin{aligned}
&(e_1)a = \sum_{i=3}^{m} e_i + \theta e_2 + e_1 + \underline{x_1 - \alpha f_1}; && (f_1)a = f_1; \\
&(e_2)a = e_2; && (f_2)a = f_2 - \theta^{-1} f_1; \\
&(e_i)a = e_i; && (f_i)a = f_i - f_1; \qquad i \in \{3, \ldots, m\} \\
&\underline{(x_1)a = x_1 - f_1},
\end{aligned} \tag{4.11}$$

and a stabilises all other vectors from β_1. Here the underlined part is in the formula only if $\zeta = 1$ and x_1 is as in Remark 4.8. In other words, if $n_{k+i} > 1$ for all $i = 1, \ldots, l$, then we omit the underlined part. It is routine to check that $\det(a) = 1$ and a is an isometry of $(V, \mathbf{f})$, so $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$.

We claim that $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q))$. Let $\varphi = \phi^j g \in \tilde{S} \cap S^a$, where g is as in (4.9). Observe that S stabilises $\langle e_1 \rangle$, so S^a stabilises $\langle e_1 \rangle a$. Therefore,

$$((e_1)a)\phi^j g = \begin{cases} \sum_{i=3}^{m} \alpha_i e_i + \theta^{p^j} \alpha_2 e_2 + \alpha_1 e_1 + \underline{\delta_1 x_1 - \alpha_1^{\dagger} \alpha f_1} \\ \lambda (e_1)a \end{cases} \tag{4.12}$$

for some $\lambda \in \mathbb{F}_{q^2}^*$. Thus,

$$\lambda = \alpha_1 = \theta^{p^j - 1} \alpha_2 = \underline{\alpha_1^{\dagger}} = \alpha_3 = \ldots = \alpha_m = \underline{\delta_1}$$

and $\theta^{p^j - 1} = 1$, so $j = 0$ and $\varphi = g \in Z(\mathrm{GU}_n(q, \mathbf{f}_\beta))$.

Let $k = 1$. We can assume that $n_{k+1} \geq 2$. Indeed, if $n_{k+i} = 1$ for all $i \in \{1, \ldots, l\}$, then, by Remark 4.6, $l = 1$, so $n = 3$ and Theorem B1 follows by Lemma 4.2. Thus, $\langle e_2, f_2 \rangle \subseteq U_1$ and $\alpha_2 = \alpha_2^{\dagger}$. We claim that there exists $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ such that

$$\begin{aligned}
&(e_1)a = \sum_{i=3}^{m} e_i + \theta e_2 + e_1 + f_2 - \theta f_1 + \underline{x_1 - \alpha f_1}; \\
&(f_1)a = f_1; \\
&(e_2)a = e_2 - f_1; \\
&(f_2)a = f_2 - \theta^q f_1; \\
&(e_i)a = e_i; && i \in \{3, \ldots, m\} \\
&(f_i)a = f_i - f_1; && i \in \{3, \ldots, m\} \\
&\underline{(x_1)a = x_1 - f_1},
\end{aligned} \tag{4.13}$$

and a stabilises all other vectors from β_1. Here the underlined part is in the formula only if $\zeta = 1$ and x_1 is as in Remark 4.8. It is routine to check that $\det(a) = 1$ and a is an isometry of $(V, \mathbf{f})$, so $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$. The arguments similar to those after (4.11) show that $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q))$.

Case (1.2). Assume that $l = 0$ (so there is no U_i) and there exists $r \in \{1, \ldots, k\}$ such that $\dim W_r \geq 4$. So

$$W_r = \langle f_s, \ldots, f_{s+n_r}, e_s, \ldots, e_{s+n_r} \rangle, \text{ for some } s \text{ and } n_r \geq 2.$$

In particular, $\alpha_s = \alpha_{s+1}$. Let $\chi = \eta^{(q+1)/2}$, so $\chi + \chi^q = 0$ and $\chi^{-q} = -\chi^{-1}$. We claim that there exists $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ such that

$$\begin{aligned}
(e_s)a &= f_s + \theta f_{s+1} + \sum_{i \notin \{s,s+1\}}^{m} f_i + \chi e_s; & (f_s)a &= -\chi^{-1} f_s;\\
(e_{s+1})a &= \chi e_{s+1} + \theta^q f_s; & (f_{s+1})a &= -\chi^{-1} f_{s+1};\\
(e_i)a &= e_i + \chi^{-1} f_s; & (f_i)a &= f_i \qquad \text{for } i \neq s.
\end{aligned}$$

It is routine to check that $\det(a) = 1$ and a is an isometry of $(V, \mathbf{f})$, so $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$.

We claim that $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q))$. Let $\varphi = \phi^j g \in \tilde{S} \cap S^a$, where g is as in (4.9). Notice that S stabilises $E = \langle e_1, \dots, e_m \rangle$, so S^a stabilises Ea. Therefore,

$$((e_s)a)\varphi = \begin{cases} \alpha_s^{\dagger} f_s + \theta^{p^j} \alpha_{s+1}^{\dagger} f_{s+1} + \sum_{i \notin \{s,s+1\}}^{m} \alpha_i^{\dagger} f_i + \chi^{p^j} \alpha_s e_s \\ \eta_1(e_1)a + \dots + \eta_m(e_m)a. \end{cases} \tag{4.14}$$

Since $((e_s)a)g$ does not have terms with e_i for $i \neq s$ in the first line of (4.14), $((e_s)a)g = \eta_s(e_s)a$, so

$$\eta_s = \chi^{p^j-1} \alpha_s = \alpha_s^{\dagger} = \theta^{p^j-1} \alpha_{s+1}^{\dagger} = \alpha_1^{\dagger} = \dots = \alpha_{s-1}^{\dagger} = \alpha_{s+1}^{\dagger} = \dots = \alpha_m^{\dagger} \tag{4.15}$$

and $\theta^{p^j-1} = 1$. Hence $j = 0$ and $\alpha_s = \alpha_s^{\dagger}$ by (4.15), so g is scalar and $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q, \mathbf{f}_{\beta_1}))$.

Case (1.3). Assume $l > 0$ and there exists $i \in \{1, \dots, k\}$ such that $\dim W_i \geq 4$. So

$$W_i = \langle f_s, \dots, f_{s+n_i}, e_s, \dots, e_{s+n_i} \rangle, \text{ for some } s \text{ and } n_i \geq 2.$$

In particular, $\alpha_s = \alpha_{s+1}$. Let $r = n_1 + \dots + n_k$. We claim that there exists $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ such that

$$(e_s)a = \Big(\sum_{i=r+1}^{m} e_i\Big) + e_s + \theta f_{s+1} + \sum_{i \notin \{s,s+1\}}^{r} f_i + \underline{(x_1 - \alpha f_s)}; \qquad (f_s)a = f_s;$$

$$\begin{aligned}
(e_{s+1})a &= e_{s+1} - \theta^q f_s; & (f_{s+1})a &= f_{s+1};\\
(e_i)a &= e_i - f_s; & (f_i)a &= f_i; & &\text{for } i \in \{1, \dots, r\} \setminus \{s, s+1\};\\
(e_i)a &= e_i; & (f_i)a &= f_i - f_s; & &\text{for } i \in \{r+1, \dots, m\};\\
&\underline{(x_1)a = x_1 - f_s,}
\end{aligned}$$

and a stabilises all other vectors from β_1. Here the underlined part is in the formula only if $\zeta = 1$ and x_1 is as in Remark 4.8. It is routine to check that $\det(a) = 1$ and a is an isometry of $(V, \mathbf{f})$, so $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$. The arguments similar to ones in **Case (1.2)** show that $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q))$.

Case 2. Let $\mu = 0$ and $k = 0$. So S stabilises no non-zero singular subspace. Choose U to be one of the U_i such that $\dim U = \max_{i \in \{1, \dots, l\}} \{\dim U_i\}$. Therefore, $V = U \perp U^{\perp}$ and $(U^{\perp})S = U^{\perp}$. Without loss of generality, we can assume that

$$U = U_1 = \langle f_1, \dots, f_d, \underline{x}, e_d, \dots, e_1 \rangle, \tag{4.16}$$

where $d = [\dim U/2]$ and $\{\underline{x}\} = \{x_1, \dots, x_t\} \cap U$. If $\dim U$ is even, then $\{\underline{x}\}$ is empty and we read (4.16) without $\underline{x}$. If $\dim U$ is odd, then we assume that $\underline{x} = x_t$. So

$$U^{\perp} = \langle f_{d+1}, \dots, f_m, x_1, \dots, x_s, e_m, \dots, e_{d+1} \rangle,$$

where $s = t-1$ if $\dim U$ is odd and $s = t$ otherwise. Define x_1 as in Remark 4.8. Notice that if $\phi^j g \in \tilde{S}$, so g has shape (4.9), then $\alpha_i^\dagger = \alpha_i$ for $i \in \{1,\dots,m\}$ since g acts on U_r containing e_i and f_i as a scalar.

If $\dim U_i = 1$ for all $i = 1,\dots,l$, then M is abelian and, by Theorem 2.10, there exists $y \in \mathrm{SU}_n(q)$ such that $M \cap M^y \leq Z(\mathrm{GU}_n(q))$. Thus, $(S \cap S^y)/Z(\mathrm{GU}_n(q))$ is an abelian subgroup of $(S \cdot \mathrm{SU}_n(q))/Z(\mathrm{GU}_n(q))$ and, by Theorem 2.10, there is $z \in \mathrm{SU}_n(q)$ such that $(S \cap S^y) \cap (S \cap S^y)^z = Z(\mathrm{GL}_n(q))$. So we can assume $\dim U \geq 2$.

The two subcases we consider correspond to the following situations: when $d = m$ and $d < m$ respectively.

Case (2.1). Let $d = m$, so $V = U \perp \langle x_1 \rangle$.

Assume $d \geq 2$, so $\alpha_1 = \alpha_2$. We claim that there exists $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ such that

$$\begin{aligned} (e_1)a &= e_1; & (f_1)a &= f_1 + (x_1 - \alpha e_1); \\ (e_2)a &= e_2; & (f_2)a &= f_2 + \theta^q (x_1 - \alpha e_2); \\ (x_1)a &= x_1 - e_1 - \theta e_2, & & \end{aligned}$$

and a stabilises all other vectors from β_1. It is routine to check that $\det(a) = 1$ and a is an isometry of $(V, \mathbf{f})$, so $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$.

We claim that $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q))$. Let $\varphi = \phi^j g \in \tilde{S} \cap S^a$, where g is as in (4.9). Observe that S stabilises $\langle x_1 \rangle$, so S^a stabilises $\langle x_1 \rangle a$. Therefore,

$$((x_1)a)\varphi = \delta_1 x_1 + \alpha_1 e_1 + \theta^{p^j} \alpha_2 e_2 = \lambda((x_1)a)$$

for some $\lambda \in \mathbb{F}_{q^2}^*$. Hence $\lambda = \delta_1 = \alpha_1 = \theta^{p^j - 1}\alpha_2$, g is scalar and $j = 0$ since $\alpha_1 = \alpha_2$. So $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q, \mathbf{f}_{\beta_1}))$.

Assume that $d = 1$, so $\dim U \leq 3$. If $\dim U = 2$, then $n = 3$ and this case is considered in Lemma 4.2, so we may assume $\dim U = 3$. We claim that there exists $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ such that

$$\begin{aligned} (e_1)a &= e_1 + x_1 - \alpha f_1; & (f_1)a &= \alpha f_1 - e_1 - \theta \underline{x}; \\ (\underline{x})a &= \theta^q e_1 + \theta^q \alpha^q f_1; & (x_1)a &= e_1 - \alpha f_1 + x_1 + \theta \underline{x}. \end{aligned}$$

It is routine to check that $\det(a) = 1$ and a is an isometry of $(V, \mathbf{f})$, so $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$. We claim that $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q))$. Let $\varphi = \phi^j g \in \tilde{S} \cap S^a$, where g is as in (4.9). Observe that S stabilises $\langle x_1 \rangle$, so S^a stabilises $\langle x_1 \rangle a$. Therefore,

$$((x_1)a)\varphi = \alpha_1 e_1 - \alpha^{p^j} \alpha_1 f_1 + \delta_1 x_1 + \alpha_1 + \theta^{p^j} \alpha_1 \underline{x} = \lambda((x_1)a)$$

for some $\lambda \in \mathbb{F}_{q^2}^*$. Hence $\lambda = \delta_1 = \alpha_1 = \theta^{p^j - 1}\alpha_1$, g is scalar and $j = 0$. So $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q, \mathbf{f}_{\beta_1}))$.

Case (2.2). Let $d < m$.

Assume $d \geq 2$, so $\alpha_1 = \alpha_2$. We claim that there exists $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ such that

$$\begin{aligned} (e_1)a &= \Big(\sum_{i=d+1}^{m} e_i\Big) + e_1 + \underline{(x_1 - \alpha f_1)}; \\ (f_1)a &= f_1; \\ (e_2)a &= \Big(\sum_{i=d+1}^{m} \theta e_i\Big) + e_2; \end{aligned}$$

$$
\begin{aligned}
&(f_2)a = f_2; && \\
&(e_i)a = e_i; && \text{for } i \in \{3, \dots, d\}; \\
&(f_i)a = f_i; && \text{for } i \in \{3, \dots, d\}; \\
&(e_i)a = e_i; && \text{for } i \in \{d+1, \dots, m\}; \\
&(f_i)a = f_i - f_1 - \theta^q f_2; && \text{for } i \in \{d+1, \dots, m\}; \\
&\underline{(x_1)a = x_1 - f_1}, &&
\end{aligned}
$$

and a stabilises all other vectors from β_1. It is routine to check that $\det(a) = 1$ and a is an isometry of $(V, \mathbf{f})$, so $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$. If $d = 1$, then let $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ be defined by (4.13). The arguments similar to ones in **Case (2.1)** applied to $((e_1)a)\varphi$ and $((e_2)a)\varphi$ show that $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q))$.

Case 3. Let $\mu = 1$, so $q \in \{2, 3, 5\}$. Without loss of generality, we can assume that $\gamma_{k+l}(M) \in \{\mathrm{GU}_2(q), \mathrm{GU}_3(2), \mathrm{MU}_4(2)\}$. Let $\{v_1, \dots, v_\nu\}$ be an orthonormal basis of U_l, so $2 \leq \nu \leq 4$. For the remaining W_i and U_i we change basis as in (4.7), so

$$\beta_1 = \{f_1, \dots, f_m, x_1, \dots, x_t, v_1, \dots, v_\nu, e_m, \dots, e_1\}, \tag{4.17}$$

and $n = 2m + t + \nu$. Denote $n_1 + \ldots + n_k$ by r, so $r \leq m$. Notice that the subspace

$$E = \langle e_1, \dots, e_r \rangle$$

is S-invariant. Let $\varphi \in \tilde{S}$, so, by **Step 1**, $\varphi = \phi^j g$ with $j \in \{0, 1\}$ and

$$
\begin{aligned}
&(e_i)g = \alpha_i e_i && \text{for } i \in \{1, \dots, m\}; \\
&(f_i)g = \alpha_i^\dagger f_i && \text{for } i \in \{1, \dots, m\}; \\
&(x_i)g = \delta_i x_i && \text{for } i \in \{1, \dots, t\}; \\
&(v_i)g = \lambda_{i1} v_1 + \ldots + \lambda_{i\nu} v_\nu && \text{for } i \in \{1, \dots, \nu\},
\end{aligned}
$$

for α_i, δ_i, $\lambda_{ji} \in \mathbb{F}_{q^2}$. Let $\alpha \in \mathbb{F}_{q^2}^*$ be such that $\alpha + \alpha^q = 1$ and $\alpha \notin \mathbb{F}_q$. It is easy to verify existence of such α for $q \in \{2, 3, 5\}$ by computation.

The three subcases we consider correspond to the following situations: when $m = r > 0$, $m > r > 0$, and $m \geq r = 0$ respectively. We exhibit $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ such that $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q, \mathbf{f}_{\beta_1}))$ in each subcase. We provide detailed proof only for **Case (3.3)**.

Case (3.1). Let $m = r > 0$, so $l \leq 2$ and $l = 2$ if and only if $\dim U_1 = 1$, so $\zeta = 1$ and $U_1 = \langle x_1 \rangle$. If $n \geq 3\nu + 1$, then $r \geq \nu$ (recall that $2 \leq \nu \leq 4$). For smaller n, Theorem B1 is verified by computation, so we assume $r \geq \nu$.

We claim that there exists $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ such that

$$
\begin{aligned}
&(e_1)a = \sum_{s=2}^{r} f_s + e_1 + (v_1 - \alpha f_1) + \underline{x_1 - \alpha f_1}; && (f_1)a = f_1 && \\
&(e_i)a = e_i - f_1 + (v_i - \alpha f_i); && (f_i)a = f_i; && i \in \{2, \dots, \nu\} \\
&(e_i)a = e_i - f_1; && (f_i)a = f_i; && i \in \{\nu+1, \dots, r\} \\
&\underline{(x_1)a = x_1 - f_1}; && (v_i)a = v_i - f_i; && i \in \{1, \dots, \nu\}
\end{aligned}
$$

and a stabilises all other vectors in β_1. Here the underlined part is in the formula only if $\zeta = 1$ and x_1 is as in Remark 4.8. It is routine to check that $\det(a) = 1$ and a is an isometry of $(V, \mathbf{f})$, so $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$. Moreover, $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q, \mathbf{f}_{\beta_1}))$.

Case (3.2). Assume that $m > r > 0$. Recall $\mu = 1$ and Remark 4.6; thus, if $\nu = 2$, then $m \geq \nu$ for $n \geq 6$; if $\nu \in \{3, 4\}$, then $m \geq \nu$ for $n \geq 9$. For smaller n, Theorem B1 is verified by computation, so we assume $m \geq \nu$.

We claim that there exists $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ such that

$$\begin{aligned}
(e_1)a &= \sum_{s=n_1+1}^{m} e_s + e_1 + (v_1 - \alpha f_1) + \underline{(x_1 - \alpha f_1)}; && \\
(f_1)a &= f_1; && \\
(e_i)a &= e_i + (v_i - \alpha f_i); && i \in \{2, \ldots, \nu\} \\
(f_i)a &= f_i; && i \in \{2, \ldots, \nu\} \\
(e_i)a &= e_i; && i \in \{\nu + 1, \ldots, m\} \\
(f_i)a &= f_i - \delta_{(i>n_1)} f_1; && i \in \{\nu + 1, \ldots, m\} \\
&\underline{(x_1)a = x_1 - f_1}; && \\
(v_i)a &= v_i - f_i; && i \in \{1, \ldots, \nu\}
\end{aligned}$$

and a stabilises all other vectors from β_1. Here the underlined part is in the formula only if $\zeta = 1$ and x_1 is as in Remark 4.8. It is routine to check that $\det(a) = 1$ and a is an isometry of $(V, \mathbf{f})$, so $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$. Moreover, $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q, \mathbf{f}_{\beta_1}))$.

Case (3.3). Assume that $r = 0$. Recall $\mu = 1$ and Remark 4.6; thus, if ν is 2, 3 or 4, then $m \geq \nu$ for $n \geq 6$, 10 and 12 respectively. For smaller n, Theorem B1 is verified by computation, so we assume $m \geq \nu$. Let U be one of $\{U_1, ..., U_{l-1}\}$ with maximum dimension. So we can assume

$$U = U_1 = \langle f_1, \ldots, f_d, \underline{x}, e_d, \ldots, e_1 \rangle$$

where d and $\underline{x}$ are defined as in (4.16).

Assume $d = 1$. We claim that there exists $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ such that

$$(e_1)a = \sum_{s=3}^{m} e_s + \alpha^\dagger e_2 + e_1 + f_2 - \alpha^\dagger f_1 + (v_1 - \alpha f_1) + \underline{(x_1 - \alpha f_1)}; \qquad (f_1)a = f_1;$$

$$\begin{aligned}
&(e_2)a = e_2 + (v_2 - \alpha f_2); && (f_2)a = f_2 - \alpha^{-1} f_1; && \\
&(e_i)a = e_i + v_i - \alpha f_i + \alpha f_1; && (f_i)a = f_i - f_1; && i \in \{3, \ldots, \nu\} \\
&(e_i)a = e_i; && (f_i)a = f_i - f_1; && i \in \{\nu + 1, \ldots, m\} \\
&\underline{(x_1)a = x_1 - f_1}; && (v_i)a = v_i - f_i; && i \in \{1, \ldots, \nu\}
\end{aligned}$$

and a stabilises all other vectors from β_1. Here the underlined part is in the formula only if $\zeta = 1$ and x_1 is as in Remark 4.8. It is routine to check that $\det(a) = 1$ and a is an isometry of $(V, \mathbf{f})$, so $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$. We claim that $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q))$. Let $\varphi \in \tilde{S} \cap S^a$.

Observe that S stabilises U, so S^a stabilises Ua. Therefore,

$$((e_1)a)\varphi = \begin{cases} \displaystyle\sum_{s=3}^{m} \alpha_s e_s + (\alpha^\dagger)^{p^j} \alpha_2 e_2 + \alpha_1 e_1 + \alpha_2 f_2 - (\alpha^\dagger)^{p^j} \alpha_1 f_1 + \\ \qquad\qquad\qquad\qquad \displaystyle +(\sum_{i=1}^{\nu} \lambda_{1i} v_i) - \alpha_1 \alpha^{p^j} f_1) + \underline{\delta_1 x_1 - \alpha_1 \alpha^{p^j - 1} f_1} \\ \eta_1 (e_1)a + \mu_{r+1}(f_{r+1}) + \lambda \underline{x}. \end{cases}$$

Thus,

$$\begin{cases}\eta_1 = \alpha_1 = \alpha_2 = (\alpha^\dagger)^{p^j-1}\alpha_2 = \alpha_3 = \ldots = \alpha_m = \lambda_{11} = \underline{\delta_1} \\ \lambda_{12} = \ldots = \lambda_{1\nu} = 0.\end{cases}$$

Hence $\alpha^{p^j-1} = 1$ and $j = 0$. The same argument for $((e_i)a)g$ with $i = r+2, \ldots, r+\nu$ shows that $\lambda_{ii} = \alpha_1$ for $i \in \{1, \ldots, \nu\}$ and $\lambda_{ij} = 0$ for $i \neq j$. Therefore, g is scalar by Remark 4.8.

Assume $d \geq 2$. We claim that there exists $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$ such that

$$\begin{aligned}
&(e_1)a = \sum_{s=d+1}^{m} e_s + e_1 + (v_1 - \alpha f_1) + \underline{(x_1 - \alpha f_1)}; && (f_1)a = f_1;\\
&(v_1)a = v_1 - f_1 - \alpha^q f_2;\\
&(e_2)a = e_2 + \alpha v_1 - \alpha f_1 + v_2 - 2\cdot\alpha f_2; && (f_2)a = f_2;\\
&(v_2)a = v_2 - f_2;
\end{aligned}$$

$$\begin{aligned}
&(e_i)a = e_i + v_i - \alpha f_i + \delta_{i>d}\alpha f_1; && (f_i)a = f_i - \delta_{i>d} f_1; && i \in \{3, \ldots, \nu\}\\
&(e_i)a = e_i; && (f_i)a = f_i - \delta_{i>d} f_1; && i \in \{\nu+1, \ldots, m\}\\
&\underline{(x_1)a = x_1 - f_1}; && (v_i)a = v_i - f_i + \delta_{i>d} f_1; && i \in \{3, \ldots, \nu\}
\end{aligned}$$

and a stabilises all other vectors from β_1. Here the underlined part is in the formula only if $\zeta = 1$ and x_1 is as in Remark 4.8; $\delta_{i>d}$ is 1 if $i > d$ and 0 otherwise. It is routine to check that $\det(a) = 1$ and a is an isometry of $(V, \mathbf{f})$, so $a \in \mathrm{SU}_n(q, \mathbf{f}_{\beta_1})$. We claim that $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q))$. Let $\varphi \in \tilde{S} \cap S^a$.

Observe that S stabilises U, so S^a stabilises Ua. Therefore, $((e_1)a)\varphi$ is

$$\sum_{s=d+1}^{m} \alpha_s e_s + \alpha_1 e_1 + (\sum_{i=1}^{\nu} \lambda_{1i} v_i) - \alpha_1 \alpha^{p^j} f_1 + \underline{\delta_1 x_1 - \alpha_1 \alpha^{p^j-1} f_1}$$

and

$$((e_1)a)\varphi = \sum_{i}^{d} \eta_i (e_i)a + \sum_{i}^{d} \mu_i (f_i) + \lambda \underline{x}.$$

Since $\eta_i(e_i)a$ for $i > 1$ has η_i as a coefficient for e_i with respect to β_1 and $((e_1)a)\varphi$ has 0 as these coefficients in the first line of the formula above, $\eta_i = 0$ for all $i > 1$. The same arguments for μ_i and λ shows that $\lambda = 0$ and $\mu_i = 0$ for $i > 1$. Thus,

$$\begin{cases}\eta_1 = \alpha_1 = \alpha_{d+1} = \ldots = \alpha_m = \lambda_{11} = \underline{\delta_1} \\ \lambda_{12} = \ldots = \lambda_{1\nu} = 0.\end{cases}$$

So g stabilises $\langle v_1 \rangle$ and its orthogonal complement $\langle v_2, \ldots, v_\nu \rangle$ in $\langle v_1, \ldots, v_\nu \rangle$. In particular $\lambda_{i1} = 0$ for $i \in \{2, \ldots, \nu\}$. Recall that $\alpha_1 = \alpha_2$, since $d \geq 2$. Consider

$$((e_2)a)\varphi = \begin{cases}\alpha_1 e_2 + \alpha^{p^j}\lambda_{11} - \alpha^{p^j}\alpha_1 f_1 + (\sum_{i=2}^{\nu} \lambda_{2i} v_i) - 2\cdot\alpha_1\alpha^{p^j} f_1 \\ \sum_i^d \eta_i (e_i)a + \sum_i^d \mu_i (f_i) + \lambda \underline{x}.\end{cases}$$

The same arguments as above show that

$$\begin{cases}\alpha_1 = \alpha^{p^j-1}\lambda_{11} = \lambda_{22} \\ \lambda_{23} = \ldots = \lambda_{2\nu} = 0.\end{cases}$$

Hence $\alpha^{p^j-1} = 1$ and $j = 0$. The same argument for $((e_i)a)g$ with $i = 3, \dots, \nu$ shows that $\lambda_{ii} = \alpha_1$ for $i \in \{1, \dots, \nu\}$ and $\lambda_{ij} = 0$ for $i \neq j$. Therefore, g is scalar and $\tilde{S} \cap S^a \leq Z(\mathrm{GU}_n(q))$.

Hence in all cases there exist four conjugates of S in G which intersect in a group of scalars, except when

$$(n, q) = (5, 2) \text{ with } k = l = 1, n_1 = 1, n_2 = 3.$$

Here $b_S(S \cdot \mathrm{SU}_n(q)) = 5$ and $\mathrm{Reg}_S(S \cdot \mathrm{GU}_n(q), 5) \geq 5$ are verified by computation. This already arises in **Case (3.1)**. This concludes the proof of Theorem 4.5. □

Theorem B1 now follows by Lemma 4.2 and Theorems 4.4 and 4.5.

5. General case: symplectic groups

We prove Theorems C1 and C2 in Sections 5.1 and 5.2 respectively.

5.1. **Solvable subgroups contained in $\Gamma\mathrm{Sp}_n(q)$.** As we mentioned in the introduction, it so more convenient to work with $\Gamma\mathrm{Sp}_n(q)$ instead of $\mathrm{P\Gamma Sp}_n(q)$ to prove Theorem C1. So, in this section S is a maximal solvable subgroup of $\Gamma\mathrm{Sp}_n(q)$ where $n \geq 4$. Our goal is to prove Theorem C1 which we reformulate in the following equivalent way.

Theorem C1. *Let $n \geq 4$. If S is a maximal solvable subgroup of $\Gamma\mathrm{Sp}_n(q)$, then $b_S(S \cdot \mathrm{Sp}_n(q)) \leq 4$, so $\mathrm{Reg}_S(S \cdot \mathrm{Sp}_n(q), 5) \geq 5$.*

If $(n, q) = (4, 2)$, then Theorem C1 is verified by computation.

Lemma 5.1. *Let Let $n \geq 2$ and $q = p^f$ where p is prime and $f > 1$. Denote $M = S \cap \mathrm{GSp}_n(q)$. If S stabilises no non-zero proper subspace of V, then there exist $y, z \in \mathrm{Sp}_n(q)$ such that $M \cap M^y \cap M^z \leq Z(\mathrm{GSp}_n(q))$.*

Proof. If $M \leq \mathrm{GSp}_n(q)$ is irreducible, then such y, z exist by Theorem 3.22.

Assume that M is reducible. Hence there exists $0 < U_1 < V$ of dimension m such that $(U_1)M = U_1$ and U_1 is $\mathbb{F}_q[M]$-irreducible. Let $\varphi \in S$ be such that $M\varphi$ is a generator of S/M, so $|M\varphi| = |S : M| = r$. Let $U_i = U_1\varphi^{i-1}$ for $i \in \{1, \dots, r\}$. By Lemma 2.5, M is completely reducible stabilising each subspace of the decomposition

$$V = U_1 \oplus \ldots \oplus U_k; \; k = n/m.$$

In particular, φ permutes the U_i cyclically and k divides $|S : M|$.

If $m = 1$, then M is abelian and the lemma follows by Theorem 2.10. So further we assume $m \geq 2$. Let us fix q and consider a minimal counterexample (n, S). So n is the smallest integer such that $\mathrm{GSp}_n(q)$ is not solvable and $\Gamma\mathrm{Sp}_n(q)$ has a maximal solvable subgroup S stabilising no non-zero proper subspaces of V, $M = S \cap \mathrm{GSp}_n(q)$ is reducible and $b_M(M \cdot \mathrm{Sp}_n(q)) > 3$.

If V is $\mathbb{F}_q[M]$-homogeneous, then M stabilises each of the V_i in a decomposition $V = V_1 \oplus \ldots \oplus V_k$ as in Lemma 2.6 and all the V_i are irreducible $\mathbb{F}_q[M]$-submodules of V by [1, (5.2) and (5.3)]. Let H_i be the restriction of M on V_i. The lemma now follows by the proof of Theorem 3.22.

Now assume that V is not $\mathbb{F}_q[M]$-homogeneous, so, by Lemma 2.6, S stabilises a decomposition of $V = V_1 \oplus \ldots \oplus V_k$ as in Lemma 2.6 and all the V_i are $\mathbb{F}_q[M]$-submodules of V.

First, assume that $k = 2$ and the V_i are totally isotropic. Hence there exist $y_1, z_1 \in \mathrm{SL}(V_1)$ such that $H_1 \cap H_1^{y_1} \cap H_1^{z_1} \leq Z(\mathrm{GL}(V_1))$ by Theorem 2.11 and Lemma 2.12. Now $y, z \in \mathrm{Sp}_n(q)$ such that $M \cap M^y \cap M^z \leq Z(\mathrm{GSp}_n(q))$ exist by the proof of Theorem 3.22 (**Case 2**).

Now assume that either $k \geq 2$ and the V_i are non-degenerate or $k \geq 4$ and the V_i are totally isotropic. Hence S stabilises the decomposition (and M stabilises each of its summands)

$$V = W_1 \perp \ldots \perp W_t$$

where $t = k$ and $W_i = V_i$ if the V_i are non-degenerate and $t = k/2$ and $W_i = V_{2i-1} \oplus V_{2i}$ otherwise. Since S stabilises no non-zero proper subspaces of V, the stabiliser of W_i in S induces a subgroup $S_i \leq \Gamma\mathrm{Sp}_{n/t}(q)$ that stabilises no non-zero proper subspaces of W_i. Let H_i be $S_i \cap \mathrm{GSp}_{n/t}(q)$. Note that since $f > 1$, the situation where $n/t = 2$ and $q \in \{2, 3, 5\}$ is not possible. So, by Theorem 3.22 and since (n, S) is a minimal counterexample, there exist $y_i, z_i \in \mathrm{Sp}_{n/t}(q)$ such that

$$H_i \cap H_i^{y_i} \cap H_i^{z_i} \leq Z(\mathrm{GSp}_{n/t}(q)).$$

Hence $y, z \in \mathrm{Sp}_n(q)$ such that $M \cap M^y \cap M^z \leq Z(\mathrm{GSp}_n(q))$ exist by the proof of Theorem 3.22 (**Case 1**). □

Theorem 5.2. *Theorem* C1 *holds if S stabilises no non-zero proper subspaces of V.*

Proof. If $f = 1$, then the theorem follows by Theorem 3.22, so we assume $f > 1$. It follows by [12, Theorem 1.1] unless S lies in a maximal subgroup H of $S \cdot \mathrm{Sp}_n(q)$ such that the action of $S \cdot \mathrm{Sp}_4(q)$ on right cosets of H is a standard action. Hence one of the following holds (see [12, Definition 2.1] and [12, Table 1]):

(*a*) $q = 2^f$ and H is of type $O_n^{\epsilon}(q)$;
(*b*) $n = 4$ and H is the stabiliser of a decomposition $V = V_1 \perp V_2$ with non-degenerate V_i of dimension 2;
(*c*) $n = 4$ and H is the normaliser in $\Gamma\mathrm{Sp}_4(q)$ of a field extension of the field of scalar matrices.

First, assume that (*a*) holds, so $q = 2^f$ with $f > 1$ and H is a group of semisimilarities of V with respect to a non-degenerate quadratic form $Q : V \to V$. Let $\mathbf{f}_Q$ be defined by

$$\mathbf{f}_Q(u, v) = Q(u + v) - Q(u) - Q(v) \text{ for all } u, v \in V. \tag{5.1}$$

By [30, Table 4.8.A], $\mathbf{f}_Q = \mathbf{f}$. By [30, Proposition 2.5.3], there exists a basis

$$\beta = \{f_1, \ldots, f_m, e_1, \ldots, e_m\}$$

as in Lemma 2.2 such that

- if $\epsilon = +$, then $Q(f_i) = Q(e_i) = 0$ for $i \in \{1, \ldots, m\}$;
- if $\epsilon = -$, then $Q(f_i) = Q(e_i) = 0$ for $i \in \{1, \ldots, m-1\}$, $Q(f_m) = \mu$ and $Q(e_m) = 1$.

Here $\mu \in \mathbb{F}_q^*$ is such that the polynomial $x^2 + x + \mu$ is irreducible over $\mathbb{F}_q$.

By Theorem 3.22, there exist $x, y \in \mathrm{Sp}_n(q)$ such that

$$S \cap S^x \cap S^y \cap \mathrm{GSp}_n(q) \leq Z(\mathrm{GSp}_n(q)).$$

Therefore, by Lemma 2.9, we may assume that if $\varphi \in S \cap S^x \cap S^y$, then $\varphi = (\phi_\beta)^j \cdot \lambda I_n$ for some $\lambda \in \mathbb{F}_q^*$ and $j \in \{0, 1, \ldots, f-1\}$.

Let θ be a generator of $\mathbb{F}_q^*$ and let $z \in \mathrm{Sp}_n(q)$ be defined as follows:

$$(e_1)z = e_1 + \theta f_1; \qquad (f_1)z = f_1;$$
$$(e_i)z = e_i; \qquad (f_i)z = f_i \qquad \text{for } i \in \{2, \ldots, m\}.$$

Notice that H^z consists of semisimilarities of V with respect to the quadratic form Q_1 defined by the rule $Q_1(v) = Q((v)z^{-1})$ for all $v \in V$. Let us show that if $\varphi \in S \cap S^x \cap S^y$ is not a scalar, then it is not a semisimilarity with respect to Q_1. Indeed, if φ is a semisimilarity with respect to Q_1, then

$$Q_1((e_1+\theta f_1)\varphi) = \delta Q_1(e_1+\theta f_1)^{\sigma} \tag{5.2}$$

for some $\lambda \in \mathbb{F}_q^*$ and $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$. Observe

$$Q_1(e_1+\theta f_1) = Q((e_1+\theta f_1)z^{-1}) = Q(e_1) = 0.$$

On the other hand

$$\begin{aligned} Q_1((e_1+\theta f_1)\varphi) &= Q_1(\lambda(e_1+\theta^{p^j} f_1)) \\ &= \lambda^2 Q_1(e_1+\theta^{p^j} f_1) \\ &= \lambda^2 Q((e_1+\theta^{p^j} f_1)z^{-1}) \\ &= \lambda^2 Q((e_1+\theta f_1)z^{-1} + ((\theta^{p^j}-\theta)f_1)z^{-1}) \\ &= \lambda^2 Q(e_1 + (\theta^{p^j}-\theta)f_1) \\ &= \lambda^2(\theta^{p^j}-\theta). \end{aligned}$$

The last equality is obtained using (5.1). Hence (5.2) holds only if $j = 0$ and φ is scalar. Therefore, $S \cap S^x \cap S^y \cap S^z \leq Z(\mathrm{GSp}_n(q))$.

Now assume that (b) holds, so S stabilises a decomposition $V = V_1 \perp V_2$ with $V_i = \langle e_i, f_i\rangle$, where $\beta = \{e_1, f_1, e_2, f_2\}$ with e_i and f_i as in (2.4). Let $y, z \in \mathrm{Sp}_4(q)$ be as in **Case 1b** of the proof of Theorem 3.22. Denote $(V_i)y$ and $(V_i)z$ by W_i and U_i respectively for $i \in \{1, 2\}$. Let θ be a generator of $\mathbb{F}_q^*$ and let $a \in \mathrm{Sp}_4(q, \mathbf{f}_\beta)$ be

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ \hline 0 & 0 & \theta & 0 \\ 0 & 1 & 0 & \theta^{-1} \end{array}\right).$$

Consider $\varphi \in S \cap S^y \cap S^z \cap S^a$, so $\varphi = \phi^j g$ with $j \in \{0, 1, \ldots, f-1\}$ and $g \in \mathrm{GSp}_4(q, \mathbf{f}_\beta)$ by Lemma 2.7. By **Case 1b** of the proof of Theorem 3.22, φ stabilises V_i, W_i and U_i for each i. Therefore, φ stabilises $\langle e_1\rangle = V_1 \cap U_1$, $\langle f_1\rangle = V_1 \cap W_1$, $\langle e_2\rangle = V_2 \cap W_2$. So φ stabilises $\langle f_1 + e_2\rangle \subseteq (V_1)z$ and $\langle f_1 + \theta e_2\rangle \subseteq (V_1)a$. Let $(e_1)g = \lambda_1 e_1$, $(f_1)g = \lambda_2 f_1$ and $(e_2)g = \lambda_3 e_2$. Therefore, $(f_1 + e_2)\varphi = \lambda_2 f_1 + \lambda_3 e_2$ and $\lambda_2 = \lambda_3$. Also, $(f_1 + \theta e_2)\varphi = \lambda_2 f_1 + \theta^{p^j}\lambda_3 e_2$, so $\theta^{p^j - 1} = 1$ and $j = 0$. In particular, $S \cap S^y \cap S^z \cap S^a \leq M \cap M^y \cap M^z$. Therefore, φ is scalar since $M \cap M^y \cap M^z \leq Z(\mathrm{GSp}_4(q))$ by **Case 1b** of the proof of Theorem 3.22.

Finally, assume that (c) holds, so S lies in the normaliser in $\Gamma\mathrm{Sp}_4(q)$ of a field extension of the field of scalar matrices. Thus, $S \leq R = \mathrm{GL}_2(q^2) \rtimes \langle\psi\rangle$ and $M \leq \mathrm{GL}_2(q^2).2 = \mathrm{GL}_2(q^2) \rtimes \langle\psi^f\rangle$ where $\psi^2 = \phi$ and $\psi^f \in \mathrm{GSp}_4(q)$. For $q \leq 3$ the theorem is verified by computation, so we assume $q \geq 4$.

Assume that M lies in the normaliser in R of a Singer cycle of $\mathrm{GL}_2(q^2)$. By [5, Lemma 3.12], there exists $x \in \mathrm{SL}_2(q^2) \leq \mathrm{Sp}_4(q)$ such that

$$S \cap S^x \leq \mathrm{GL}_2(q^2).2 \leq \mathrm{GSp}_4(q),$$

so $S \cap S^x \leq M$. By Lemma 5.1, there exist y, z such that $M \cap M^y \cap M^z \leq Z(\mathrm{GSp}_4(q))$, so

$$(S \cap S^x) \cap S^y \cap S^z \leq Z(\mathrm{GSp}_4(q)).$$

Assume that M does not lie in the normaliser in R of a Singer cycle of $\mathrm{GL}_2(q^2)$ and let $M_1 = M \cap \mathrm{GL}_2(q)$. By Theorem 2.11, there exists $x \in \mathrm{SL}_2(q^2)$ such that $M_1 \cap M_1^x \leq Z(\mathrm{GL}_4(q))$. Hence $S \cap S^x \leq Z(\mathrm{GL}_2(q^2)) \rtimes \langle \psi \rangle$. Let N be $S \cap S^x$. So $|N/Z(\mathrm{GSp}_4(q))|$ divides $(q+1) \cdot 2f$ and

$$A = |N/Z(\mathrm{GSp}_4(q)) \cap \mathrm{PGSp}_4(q)| \tag{5.3}$$

divides $(q+1) \cdot 2f$.

We claim that $\hat{Q}((N \cdot \mathrm{Sp}_4(q)/Z(\mathrm{GSp}_4(q)), 2) < 1$ where $\hat{Q}(G,c)$ is as in (2.10). Denote $N/Z(\mathrm{GSp}_4(q))$ by H. By Lemma 2.18, if $x_1, \dots, x_k$ represent distinct G-classes such that $\sum_{i=1}^k |x_i^G \cap H| \leq A$ and $|x_i^G| \geq B$ for all $i \in \{1, \dots, k\}$, then

$$\sum_{i=1}^m |x_i^G| \cdot \mathrm{fpr}(x_i)^c \leq B \cdot (A/B)^c.$$

We take A as in (5.3) since $A \geq |H|$. The proof of Lemma 3.9 implies that $\nu(g) \geq n/2 = 2$ for $g \in N \cap \mathrm{GSp}_4(q)$. For elements in $\mathrm{PGSp}_4(q)$ of prime order with $s = \nu(x) \in \{2,3\}$ we use (2.12) as a lower bound for $|x_i^G|$. If $x \in H \backslash \mathrm{PGSp}_4(q)$ has prime order, then we use the corresponding bound for $|x^G|$ in [9, Corollary 3.49]. We take B to be the smallest of these bounds for $|x_i^G|$. Such A and B are sufficient to obtain $\hat{Q}((N \cdot \mathrm{Sp}_4(q)/Z(\mathrm{GSp}_4(q)), 2) < 1$ for $q > 4$. Hence $b_S(S \cdot \mathrm{SU}_4(q)) \leq 4$. For $q = 4$ the statement is verified by computation. □

Theorem 5.3. *Theorem* C1 *holds for $q \notin \{2,3,5\}$ if S stabilises a non-zero proper subspace of V.*

Proof. The proof proceeds in two steps. In **Step 1** we obtain three conjugates of S such that elements of their intersection have special shape. In **Step 2** we find a fourth conjugate of S such that the intersection of the four is a group of scalars.

Step 1. This is similar to the first step of the proof of Theorem B1. Fix a basis β of $(V, \mathbf{f})$ as in Lemma 2.8, so $\mathbf{f}_\beta$ is as in (2.7) and elements of S take shape $\phi^j g$ with g as in (2.8) and $j \in \{0, 1, \dots, f-1\}$. We consider S as a subgroup of $\Gamma\mathrm{Sp}_n(q, \mathbf{f}_\beta)$ and let $M = S \cap \mathrm{GSp}_n(q, \mathbf{f}_\beta)$. We obtain three conjugates of S such that their intersection consists of elements $\phi^j g$ where g is diagonal with respect to β.

Let γ_i be as in Lemma 2.8. Let x be the matrix

$$\begin{pmatrix} & & & & & & & & I_{n_1} \\ & & & & & & & \iddots & \\ & & & & & & I_{n_k} & & \\ & & & I_{n_{k+1}} & & & & & \\ & & & & \ddots & & & & \\ & & & & & I_{n_{k+l}} & & & \\ & & -I_{n_k} & & & & & & \\ & \iddots & & & & & & & \\ -I_{n_1} & & & & & & & & \end{pmatrix}. \tag{5.4}$$

Observe that $x \mathbf{f}_\beta x^\top = \mathbf{f}_\beta$, so $x \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$. It is easy to see that if $g \in M$, so it has shape (2.8), then g^x has shape (5.6).

Therefore, by Theorem 2.11 and Lemmas 2.12, 4.1 and 5.1, there exist $y_i, z_i \in \mathrm{GL}_{n_i}(q)$ for $i = 1, \dots, k$ and $y_i, z_i \in \mathrm{Sp}_{n_i}(q)$ for $i \in \{k+1, \dots, k+l\}$ such that

$$\begin{aligned} \gamma_i(M) \cap \gamma_i(M)^{y_i} \cap (\gamma_i(M)^\dagger)^{z_i} &\leq Z(\mathrm{GL}_{n_i}(q)) \text{ for } i \in \{1, \dots, k\}; \\ \gamma_i(M) \cap \gamma_i(M)^{y_i} \cap \gamma_i(M)^{z_i} &\leq Z(\mathrm{GL}_{n_i}(q)) \text{ for } i \in \{k+1, \dots, k+l\}. \end{aligned} \tag{5.5}$$

$$
\left(\begin{array}{ccc|ccc|ccc}
\gamma_1(g) & & 0 & & & & & & 0 \\
* & \ddots & & & & & & & \\
* & * & \gamma_k(g) & & & & & & \\
\hline
* & \dots & * & \gamma_{k+1}(g) & & 0 & & & \\
* & \dots & * & & \ddots & & & & \\
* & \dots & * & 0 & & \gamma_{k+l}(g) & & & \\
\hline
* & \dots & & & & * & \tau(g)\gamma_k(g)^{\dagger} & & 0 \\
* & \dots & & & & * & * & \ddots & \\
* & \dots & & & & * & * & * & \tau(g)\gamma_1(g)^{\dagger}
\end{array}\right)
\tag{5.6}
$$

Denote by y and z the block-diagonal matrices

$$
\begin{aligned}
&\mathrm{diag}[y_1^{\dagger}, \dots, y_k^{\dagger}, y_{k+1}, \dots, y_{k+l}, y_k, \dots, y_1] \text{ and} \\
&\mathrm{diag}[z_1^{\dagger}, \dots, z_k^{\dagger}, z_{k+1}, \dots, z_{k+l}, z_k, \dots, z_1]
\end{aligned}
\tag{5.7}
$$

respectively. It is routine to check that $y, z \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$.

Therefore, if $g \in M \cap M^{xz}$, then g is the block-diagonal matrix

$$
\mathrm{diag}[\tau(g)g_1^{\dagger}, \dots, \tau(g)g_k^{\dagger}, g_{k+1}, \dots, g_{k+l}, g_k, \dots, g_1],
\tag{5.8}
$$

where $g_i \in \gamma_i(M) \cap (\gamma_i(M)^{\dagger})^{z_i}$ for $i \in \{1, \dots, k+l\}$.

Thus, if $g \in M \cap M^y \cap M^{xz}$, then g has shape (5.8) where

$$
\begin{aligned}
g_i \in \gamma_i(M) \cap \gamma_i(M)^{y_i} \cap (\gamma_i(M)^{\dagger})^{z_i} &\leq Z(\mathrm{GL}_{n_i}(q)) \text{ for } i \in \{1, \dots, k\}; \\
g_i \in \gamma_i(M) \cap \gamma_i(M)^{y_i} \cap \gamma_i(M)^{z_i} &\leq Z(\mathrm{Sp}_{n_i}(q)) \text{ for } i \in \{k+1, \dots, k+l\}.
\end{aligned}
$$

In particular, g is

$$
\mathrm{diag}[\tau(g)\alpha_1^{\dagger} I_{n_1}, \dots, \tau(g)\alpha_k^{\dagger} I_{n_k}, \alpha_{k+1} I_{n_{k+1}}, \dots, \alpha_{k+l} I_{n_{k+l}}, \alpha_k I_{n_k}, \dots, \alpha_1 I_{n_1}],
\tag{5.9}
$$

where $\alpha_i \in \mathbb{F}_q$ for $i \in \{1, \dots, k+l\}$ and $\alpha_i^{\dagger} = \alpha_i^{-1}$ for $i \in \{1, \dots, k\}$. By Lemma 2.9, we can assume that elements in $\gamma_i(S) \cap \gamma_i(S)^{y_i} \cap (\gamma_i(S)^{\dagger})^{z_i}$ for $i \leq k$ and in $\gamma_i(S) \cap \gamma_i(S)^{y_i} \cap (\gamma_i(S))^{z_i}$ for $i > k$ have shape $\phi^j g_i$ with $g_i \in Z(\mathrm{GL}_{n_i}(q))$. Thus, if $\varphi \in S \cap S^y \cap S^{xz}$, then $\varphi = \phi_\beta^j g$ with g as in (5.9). Denote $S \cap S^y \cap S^{xz}$ by $\tilde{S}$.

Step 2. We now find a fourth conjugate of S such that its intersection with $\tilde{S}$ lies in $Z(\mathrm{GSp}_n(q))$.

Recall that β is such that $\mathbf{f}_\beta$ is as in (2.7). Therefore,

$$
\beta = \beta_{(1,1)} \cup \dots \cup \beta_{(1,k)} \cup \beta_{k+1} \cup \dots \cup \beta_{k+l} \cup \beta_{(2,1)} \cup \dots \cup \beta_{(2,k)},
$$

where

$$
\begin{aligned}
\beta_{(1,i)} &= \{f_1^i, \dots, f_{n_i}^i\} \text{ for } i \in \{1, \dots, k\}; \\
\beta_{(2,i)} &= \{e_1^i, \dots, e_{n_i}^i\} \text{ for } i \in \{1, \dots, k\}; \\
\beta_i &= \{f_1^i, \dots, f_{n_i/2}^i, e_1^i, \dots, e_{n_i/2}^i\} \text{ for } i \in \{k+1, \dots, k+l\},
\end{aligned}
\tag{5.10}
$$

and $(f_i^j, e_i^j) = 1$ for all i, j. All other pairs of vectors from β are orthogonal. For simplicity we relabel vectors f_i^j in β in the order they appear in β using just one index, so f_i^j becomes

$$\begin{array}{ll} f_{(\sum_{t=0}^{j-1} n_t + i)} & \text{if } j \leq k+1; \\ f_{(\sum_{t=0}^{k} n_t + \sum_{t=k+1}^{j-1}(n_t/2)+i)} & \text{if } j > k+1. \end{array}$$

We relabel the e_i^j such that $(f_i, e_i) = 1$.

If $\varphi \in \tilde{S}$, so $\varphi = \phi^j g$ with g as in (5.9), then let $\delta_i \in \mathbb{F}_q$ be such that $(e_i)g = \delta_i e_i$ for $i \in \{1, \dots, n/2\}$ (so δ_i is some α_j from (5.9)). Let θ be a generator of $\mathbb{F}_q^*$.

The remainder of the proof splits into two cases: when $k \geq 1$ and $k = 0$. In each we show that $b_S(S \cdot \mathrm{SU}_n(q)) \leq 4$.

Case 1. Let $k \geq 1$. This step splits into two subcases. In the first $n_i = 1$ for all $i \in \{1, \dots, k\}$; in the second there exists $i \in \{1, \dots, k\}$ such that $n_i \geq 2$.

Case (1.1). Let $n_i = 1$ for all $i \in \{1, \dots, k\}$. We redefine y in (5.7) to be

$$\mathrm{diag}[A^\dagger, y_{k+1}, \dots, y_{k+l}, A]$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & \ddots & & \\ 0 & \dots & 0 & 1 & 0 \\ 1 & \dots & \dots & 1 & 1 \end{pmatrix}.$$

It is easy to see that $y \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$. Let $x, z \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$ be as in **Step 1**, so $\varphi \in \tilde{S}$ has shape $\phi^j g$ with g as in (5.9). Since S stabilises $\langle e_1 \rangle$, S^y stabilises $\langle e_1 \rangle y = \langle e_1 + \dots + e_k \rangle$. Therefore,

$$((e_1)y)\varphi = (e_1 + \dots + e_k)\varphi^j g = \alpha_1 e_1 + \dots + \alpha_k e_k = \lambda(e_1 + \dots + e_k)$$

for some $\lambda \in \mathbb{F}_q^*$, so $\alpha_1 = \dots = \alpha_k$.

Assume $k \geq 2$. Consider $a \in \mathrm{GL}_n(q)$ such that

$$\begin{array}{lll} (e_1)a = \sum_{i=3}^{n/2} e_i + e_1 + \theta e_2 + f_1; & (f_1)a = f_1; & \\ (e_2)a = e_2; & (f_2)a = f_2 - \theta f_1; & \\ (e_i)a = e_i; & (f_i)a = f_i - f_1; & i \in \{3, \dots, n/2\}. \end{array}$$

It is routine to check that a is an isometry of $(V, \mathbf{f})$, so we can consider a as an element of $\mathrm{Sp}_n(q, \mathbf{f}_\beta)$. We claim that $\tilde{S} \cap S^a \leq Z(\mathrm{GSp}_n(q, \mathbf{f}_\beta))$. Let $\varphi = \phi^j g \in \tilde{S} \cap S^a$. Since S stabilises the subspace $\langle e_1 \rangle$, S^a stabilises $\langle e_1 \rangle a$. Therefore,

$$((e_1)a)\varphi = \begin{cases} \sum_{i=m+1}^{n/2} \delta_i e_i + \alpha_1 e_1 + \theta^{p^j} \alpha_1 e_2 + \tau(g)\alpha_1^\dagger f_1 \\ \lambda(e_1)a \end{cases}$$

for some $\lambda \in \mathbb{F}_q$. Hence

$$\alpha_1 = \theta^{p^j - 1}\alpha_1 = \tau(g)\alpha_1^\dagger = \delta_{m+1} = \dots = \delta_{n/2}.$$

Therefore, $j = 0$, $\alpha_1 = \dots = \alpha_{k+l}$ and $\alpha_i = \tau(g)\alpha_i^\dagger$ for all i, so g is scalar and $\tilde{S} \cap S^a \leq Z(\mathrm{GSp}_n(q, \mathbf{f}_\beta))$.

Assume $k = 1$, so $l \geq 1$ (otherwise $n = 2$) and $\{f_2, e_2\} \subseteq \beta_{k+1}$. In particular, if $\varphi = \phi^j g \in \tilde{S}$, then $(f_2)g = (e_2)g = \alpha_2$. Consider $a \in \mathrm{GL}_n(q)$ such that

$$\begin{aligned} (e_1)a &= \sum_{i=3}^{n/2} e_i + e_1 + \theta e_2 + f_2; & (f_1)a &= f_1; \\ (e_2)a &= e_2 + f_1; & (f_2)a &= f_2 - \theta f_1; \\ (e_i)a &= e_i; & (f_i)a &= f_i - f_1; \quad i \in \{3, \dots, n/2\}. \end{aligned}$$

It is routine to check that a is an isometry of $(V, \mathbf{f})$, so we can consider a as an element of $\mathrm{Sp}_n(q, \mathbf{f}_\beta)$. The arguments similar to ones for $k \geq 2$ show that $\tilde{S} \cap S^a \leq Z(\mathrm{GSp}_n(q, \mathbf{f}_\beta))$.

Case (1.2). Denote $r := \sum_{i=1}^{k} n_i$. Let $n_i \geq 2$ for some $i \leq k$, so $\delta_s = \delta_{s+1}$ for some $s < r$. Consider $a \in \mathrm{GL}_n(q)$ such that

$$\begin{aligned} (e_s)a &= f_s + \theta f_{s+1} + \sum_{i \notin \{s, s+1\}}^{n/2} f_i + e_s; & (f_s)a &= f_s; \\ (e_{s+1})a &= e_{s+1} + \theta f_s; & (f_{s+1})a &= f_{s+1}; \\ (e_i)a &= e_i + f_s; & (f_i)a &= f_i; \quad i \in \{3, \dots, n/2\} \backslash \{s, s+1\}. \end{aligned}$$

It is routine to check that a is an isometry of $(V, \mathbf{f})$, so we can consider a as an element of $\mathrm{Sp}_n(q, \mathbf{f}_\beta)$. The arguments similar to ones in **Case (1.1)** show that $\tilde{S} \cap S^a \leq Z(\mathrm{GSp}_n(q, \mathbf{f}_\beta))$.

Case 2. Let $k = 0$, so $l \geq 2$. Denote $s := n_1/2$. Hence $\{f_{s+1}, e_{s+1}\} \subseteq \beta_2$. In particular, if $\varphi = \phi^j g \in \tilde{S}$, then $(f_{s+1})g = (e_{s+1})g = \alpha_2$. Consider $a \in \mathrm{GL}_n(q)$ such that

$$\begin{aligned} (e_1)a &= \sum_{i=s+1}^{n/2} e_i + e_1 + \theta f_{s+1}; & (f_1)a &= f_1; \\ (e_{s+1})a &= e_{s+1} + \theta f_1; & (f_{s+1})a &= f_{s+1} - f_1; \end{aligned}$$

$$\begin{aligned} (e_i)a &= e_i; \quad & (f_i)a &= f_i; \quad & i &\in \{2, \dots, n_1/2\}. \\ (e_i)a &= e_i; \quad & (f_i)a &= f_i - f_1; \quad & i &\in \{s+2, \dots, n/2\}. \end{aligned}$$

It is routine to check that a is an isometry of $(V, \mathbf{f})$, so we can consider a as an element of $\mathrm{Sp}_n(q, \mathbf{f}_\beta)$. The arguments similar to ones in **Case (1.1)** show that $\tilde{S} \cap S^a \leq Z(\mathrm{GSp}_n(q, \mathbf{f}_\beta))$. $\square$

We have now proved Theorem C1 for $q \notin \{2, 3, 5\}$.

Remark 5.4. Equation (5.5) does not always hold for $q \in \{2, 3, 5\}$. In particular, it does not hold in each of the following cases:

(a) $\gamma_i(S) \leq \mathrm{GL}_2(q)$ for $i \in \{1, \dots, k\}$. Here $\gamma_i(S) = \mathrm{GL}_2(q)$ for $q \in \{2, 3\}$ and $\gamma_i(S)$ is S in (2) of Theorem 3.22 for $q = 5$.

(b) $\gamma_i(S) \leq \mathrm{GSp}_2(q)$ for $i \in \{k+1, \dots, k+l\}$. Here $\gamma_i(S) = \mathrm{GL}_2(q)$ for $q \in \{2, 3\}$ and $\gamma_i(S)$ is S in (2) of Theorem 3.22 for $q = 5$.

(c) $q \in \{2, 3\}$, $\gamma_i(S)$ is the stabiliser in $\mathrm{GSp}_4(q)$ of the decomposition $V = V_1 \perp V_2$ with V_1 and V_2 non-degenerate of dimension 2 for $i \in \{k+1, \dots, k+l\}$. Recall that $\beta_i = \{f_1^i, f_2^i, e_1^i, e_2^i\}$ and let $V_r = \langle f_r^i, e_r^i \rangle$ for $r = 1, 2$.

The following two lemmas are verified by computation. Let Q, R and T be $\gamma_i(S)$ from (a), (b) and (c) of Remark 5.4 respectively (so they depend on q).

Lemma 5.5. *Let $q \in \{2,3,5\}$, $S \leq \mathrm{Sp}_n(q)$ is a maximal solvable subgroup and β is a basis of V as in Lemma 2.8, so matrices in S have shape (2.8). Specifically, let one of the following hold:*

- *$k = 0$, $l = 2$, $q \in \{2,3,5\}$, $\gamma_i(S)$ is R for both $i = 1,2$, so $n = 4$;*
- *$k = 0$, $l = 2$, $q \in \{2,3\}$, $\gamma_i(S)$ is T for both $i = 1,2$, so $n = 8$;*
- *$k = 0$, $l = 2$, $q \in \{2,3\}$, $\gamma_1(S)$ is R, $\gamma_2(S)$ is T, so $n = 6$;*
- *$k = 1$, $l = 1$, $q \in \{2,3,5\}$, $\gamma_1(S)$ is Q, $\gamma_2(S)$ is R, so $n = 6$;*
- *$k = 1$, $l = 1$, $q \in \{2,3\}$, $\gamma_1(S)$ is Q, $\gamma_2(S)$ is T, so $n = 8$;*
- *$k = 2$, $l = 0$, $q \in \{2,3,5\}$, $\gamma_i(S)$ is Q, for both $i = 1,2$, so $n = 8$;*
- *$k = 1$, $l = 0$, $q = 5$, $\gamma_1(S)$ is Q, so $n = 4$.*

Then there exist $y, z \in \mathrm{Sp}_n(q)$ such that

$$S \cap S^y \cap S^z \leq Z(\mathrm{Sp}_n(q)).$$

Lemma 5.6. *Let $q \in \{2,3\}$, let $S \leq \mathrm{Sp}_4(q)$ be a maximal solvable subgroup and let β be a basis of V as in Lemma 2.8, so matrices in S have shape (2.8).*

(1) *Let $k = 0$, $l = 1$ and let $S = T$. If $q = 3$, then there exist $y, z \in \mathrm{Sp}_4(3)$ such that*

$$S \cap S^y \cap S^z = \left\langle 2I_4, \begin{pmatrix} 1&0&1&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} \right\rangle.$$

For instance,

$$y = \begin{pmatrix} 0&1&2&1 \\ 2&2&1&0 \\ 0&0&2&1 \\ 0&0&2&0 \end{pmatrix}, \ z = \begin{pmatrix} 0&2&0&2 \\ 0&0&2&0 \\ 0&0&1&2 \\ 1&1&1&1 \end{pmatrix}.$$

(2) *Let $k = 1$, $l = 0$ and $\gamma_1(S) = Q = \mathrm{GL}_2(q)$. If $q = 3$, then there exist $y, z \in \mathrm{Sp}_4(3)$ such that*

$$S \cap S^y \cap S^z = \left\langle \begin{pmatrix} 1&0&0&0 \\ 1&2&0&0 \\ 0&0&1&1 \\ 0&0&0&2 \end{pmatrix}, \begin{pmatrix} 2&2&0&0 \\ 2&1&0&0 \\ 0&0&2&2 \\ 0&0&2&1 \end{pmatrix} \right\rangle.$$

For instance,

$$y = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix}, \ z = \begin{pmatrix} 2&1&0&0 \\ 1&0&2&2 \\ 2&0&1&2 \\ 1&2&1&1 \end{pmatrix}.$$

(3) *If $q \in \{2,3\}$, then, in both (1) and (2), there exist $y, z, v \in \mathrm{Sp}_n(q)$ such that*

$$S \cap S^y \cap S^z \cap S^v = Z(\mathrm{Sp}_n(q)).$$

Theorem 5.7. *Theorem C1 holds for $q \in \{2,3,5\}$ if S stabilises a non-zero proper subspace of V.*

Proof. Notice that $\Gamma\mathrm{Sp}_n(q) = \mathrm{GSp}_n(q)$. As in the case $q > 3$, in **Step 1** we obtain three conjugates of S in $S \cdot \mathrm{Sp}_n(q)$ such that their intersection consists of diagonal matrices and matrices which have few non-zero entries not on the diagonal. In **Step 2** we find a fourth conjugate of S such that the intersection of the four is a group of scalars.

Step 1. We commence with a technical definition.

Definition 5.8. Let $\beta = \{v_1, \dots, v_n\}$ be a basis of a vector space V over a field $\mathbb{F}$. Let $g \in \mathrm{GL}(V)$, so $g_\beta \in \mathrm{GL}_n(\mathbb{F})$. We label the rows and columns of g_β by corresponding basis vectors, so the i-th row (column) is labelled by v_i. If $\widehat{\beta}$ is a subset of β, then the **restriction** $\widehat{g_\beta}$ of g_β to $\widehat{\beta}$ is the matrix in $\mathrm{GL}_{|\widehat{\beta}|}(\mathbb{F})$ obtained from g_β by taking only the entries lying on the intersections of rows and columns labelled by vectors in $\widehat{\beta}$. If $h \in \mathrm{GL}_{|\widehat{\beta}|}(\mathbb{F})$, then the

$(h, \widehat{\beta})$-**replacement** of g_β is the matrix obtained from g_β by replacing the entries lying on the intersections of rows and columns labelled by vectors in $\hat{\beta}$ by corresponding entries of h.

For example, if $n = 4$, $\beta = \{v_1, v_2, v_3, v_4\}$, $\widehat{\beta} = \{v_2, v_4\}$,

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} \text{ and } h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

then the restriction of g to $\widehat{\beta}$ and the $(h, \widehat{\beta})$-replacement of g are

$$\begin{pmatrix} g_{22} & g_{24} \\ g_{42} & g_{44} \end{pmatrix}, \text{ and } \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & h_{11} & g_{23} & h_{12} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & h_{21} & g_{43} & h_{22} \end{pmatrix}$$

respectively.

We claim that we can assume that there is at most one $i \in \{1, \ldots, k + l\}$ such that $\gamma_i(S)$ is one of the groups in Remark 5.4. Assume $r < s$ are the only elements of $\{1, \ldots, k + l\}$ such that $\gamma_r(S)$ and $\gamma_s(S)$ are as in Remark 5.4. Recall that β is as in (5.10). Let $\hat{\beta}$ be $\beta_r \cup \beta_s$ if $r > k$ and $\beta_{(1,r)} \cup \beta_s \cup \beta_{(2,r)}$ if $r < k$. Consider $g \in S$ and notice that its restriction $\widehat{g}$ to $\widehat{\beta}$ lies in $\mathrm{GSp}_{|\widehat{\beta}|}(q, \mathbf{f}_{\widehat{\beta}})$, where $\mathbf{f}_{\widehat{\beta}}$ is the restriction of $\mathbf{f}_\beta$ to $\langle \widehat{\beta} \rangle$ with respect to the basis $\widehat{\beta}$. If $\widehat{S}$ is the group consisting of restrictions to $\widehat{\beta}$ for all $g \in S$, then, as is easy to see, $\widehat{S}$ is one of the groups in Lemma 5.5. For example, if g is as in (2.8), $r = 1$ and $s = k + l$, then $\widehat{g}$ is constructed by the dark gray blocks of the following matrix

$$\left(\begin{array}{ccccccccc} \tau(g)\gamma_1(g)^\dagger & * & * & * & \ldots & * & * & * & * \\ & \ddots & & & & \ddots & & & \\ 0 & & \tau(g)\gamma_k(g)^\dagger & * & * & * & \ldots & * & * \\ & & & \gamma_{k+1}(g) & & 0 & * & \ldots & * \\ & & & & \ddots & & & \ddots & \\ & & & 0 & & \gamma_{k+l}(g) & * & \ldots & * \\ & & & & & & \gamma_k(g) & * & * \\ & & & & & & & \ddots & * \\ 0 & & & & & & 0 & & \gamma_1(g) \end{array}\right).$$

Let $h \in \mathrm{Sp}_{|\widehat{\beta}|}(q, \mathbf{f}_{\widehat{\beta}})$ and let t be the $(h, \widehat{\beta})$-replacement of I_n. It is routine to check that $t \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$ and the restriction of g^t to $\widehat{\beta}$ is $\widehat{g}^h$. Let x be the matrix (5.4). Notice that if $\widehat{S}$, $\widehat{S^x}$ and $\widehat{x}$ are the restrictions of S, S^x and x to $\widehat{\beta}$ respectively, then $\widehat{S^x} = \widehat{S}^{\widehat{x}}$. Therefore, by Lemma 5.5, there exist $\hat{y}, \hat{z} \in \mathrm{Sp}_{|\widehat{\beta}|}(q, \mathbf{f}_{\widehat{\beta}})$ such that

$$\widehat{S} \cap \widehat{S}^{\widehat{x}\widehat{y}} \cap \widehat{S}^{\widehat{z}} \leq Z(\mathrm{GSp}_{|\widehat{\beta}|}(q, \mathbf{f}_{\widehat{\beta}})).$$

For $i \neq r, s$ define y_i and z_i as in (5.5). Let y and z be the $(\widehat{y}, \widehat{\beta})$-replacement and $(\widehat{z}, \widehat{\beta})$-replacement of matrices from (5.7) respectively. It is routine to check that $y, z \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$. Observe now that $\tilde{S} = S \cap S^{xy} \cap S^z$ is a group of diagonal matrices.

If there is more than one such pair (r,s), then the same corrections of y and z for each pair can be done. Therefore, we can assume that there is at most one $s \in \{1,\dots,k+l\}$ such that $\gamma_s(S)$ is one of the groups in Remark 5.4. If there is no such s, then **Step 2** of the proof of Theorem 5.3 implies the theorem, so assume that such s exists. Hence, defining x to be the matrix (5.4), y and z to be as in (5.7) where y_i and z_i are as in (5.5) for $i \neq s$ and $y_s = z_s = I_{n_s}$, we obtain that $\tilde{S} = S \cap S^{xy} \cap S^z$ consists of matrices of the following shape where $\alpha_i \in \mathbb{F}_q^*$ and $\Lambda \in \gamma_s(S)$. If $s > k$ (so we may assume $s = k+1$):

$$\mathrm{diag}[\tau(g)\alpha_1 I_1, \dots, \tau(g)\alpha_k I_{n_k}, \Lambda, \alpha_{k+2} I_{n_{k+2}}, \dots, \alpha_{k+l} I_{n_{k+l}}, \alpha_k I_{n_k}, \dots, \alpha_1 I_1]. \tag{5.11}$$

If $s \leq k$:

$$\begin{aligned}\mathrm{diag}[\tau(g)\alpha_1 I_{n_1}, \dots, \tau(g)\alpha_{s-1} I_{n_{s-1}}, \tau(g)\Lambda^\dagger, \tau(g)\alpha_{s+1} I_{n_{s+1}}, \dots, \tau(g)\alpha_k I_{n_k},\\ \alpha_{k+1} I_{n_{k+1}}, \dots, \alpha_{k+l} I_{n_{k+l}},\\ \alpha_k I_{n_k}, \dots, \alpha_{s+1} I_{n_{s+1}}, \Lambda, \alpha_{s-1} I_{n_{s-1}}, \dots, \alpha_1 I_{n_1}].\end{aligned} \tag{5.12}$$

Step 2. Let $s \in \{1,\dots,k+l\}$ be such that $\gamma_s(S)$ is Q, R or T as defined after Remark 5.4.

Since the only diagonal matrix in $\mathrm{Sp}_n(2)$ is I_n, it is enough to obtain four conjugates of S in $\mathrm{Sp}_n(2)$ such that their intersection is a group of diagonal matrices. Therefore, if $\gamma_s(S) \in \{\mathrm{GL}_2(2), \mathrm{Sp}_2(2) \wr \mathrm{Sym}(2)\}$, then by Lemma 5.6 and Theorem 3.22 using the construction in **Step 1** we obtain $y, z, v \in \mathrm{Sp}_n(q)$ such that

$$S \cap S^y \cap S^z \cap S^v = \{1\}.$$

Hence for $q = 2$ we only need to consider the situation $\gamma_s(S) = R$.

We consider three distinct cases – when $\gamma_s(S)$ is R, T and Q respectively.

Case 1. First assume $\gamma_s(S)$ is R, so $s > k$. Without loss of generality, we can assume $s = k+1$. Let β be as in **Step 2** of the proof of Theorem 5.3. Let r be such that restriction of matrices from S to vectors $\{f_r, e_r\}$ is $\gamma_s(S)$. We relabel vectors in β as follows:

- f_r and e_r become f and e respectively;
- if $i < r$, then f_i and e_i remain f_i and e_i respectively;
- if $i > r$, then f_i and e_i become f_{i-1} and e_{i-1} respectively.

Therefore, since $\alpha^\dagger = \alpha$ for $\alpha \in \mathbb{F}_q$ with $q \in \{2,3,5\}$, a matrix $g \in \tilde{S}$ has shape (5.11) with

$$\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}.$$

We consider the following four subcases: $(k \geq 1, n_1 \geq 2)$, $(k = 1, n_1 = 1)$, $(k \geq 2, n_1 = 1)$, $(k = 0, n_1 \geq 1)$.

Let $k \geq 1$ and $n_1 \geq 2$. Now $W = \langle e_1, \dots, e_{n_1} \rangle$ is an S-invariant subspace. Let $m = n_1$ and let $a \in \mathrm{GL}_n(q)$ be such that

$$\begin{aligned}(e_1)a &= \sum_{i=m+1}^{(n-2)/2} e_i + e_1 + e;\\ (f_1)a &= f_1;\\ (e_2)a &= \sum_{i=m+1}^{(n-2)/2} e_i + e_2 + f - f_1;\end{aligned}$$

$$\begin{aligned}
&(f_2)a = f_2; && \\
&(e_j)a = \sum_{i=m+1}^{(n-2)/2} e_i + e_j; && j \in \{3,\dots,m\} \qquad (5.13)\\
&(f_j)a = f_j; && j \in \{3,\dots,m\}\\
&(e_j)a = e_j; && j \in \{m+1,\dots,(n-2)/2\}\\
&(f_j)a = f_j - \sum_{i=1}^{m} f_i; && j \in \{m+1,\dots,(n-2)/2\}\\
&(e)a = e + f_2; &&\\
&(f)a = f - f_1. &&
\end{aligned}$$

It is routine to check that $a \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$. Consider $g \in \tilde{S} \cap S^a$. Since S stabilises the subspace W, S^a stabilises $(W)a$. As in the proof of Theorem 5.3, let $\delta_i \in \mathbb{F}_q$ be such that $(e_i)g = \delta_i e_i$ for $i \in \{1, \dots, (n-1)/2\}$. Therefore,

$$((e_1)a)g = \begin{cases} \sum_{i=m+1}^{(n-2)/2} \delta_i e_i + \alpha_1 e_1 + \lambda_3 f + \lambda_4 e; \\ \eta_1 (e_1)a + \ldots + \eta_m (e_m)a \end{cases} \tag{5.14}$$

for some $\eta_1, \dots, \eta_m \in \mathbb{F}_q$. Observe that $((e_1)a)g$ does not have e_i for $1 < i \leq m$ in the first line of (5.14), so $((e_1)a)g = \eta_1(e_1)a$; thus $\lambda_2 = 0$ and

$$\lambda_4 = \alpha_1 = \delta_{m+1} = \ldots = \delta_{(n-2)/2}.$$

The same argument for $((e_2)a)g$ shows that $\lambda_2 = 0$ and $\tau(g)\alpha_1 = \lambda_1 = \alpha_1$. Therefore, $g = \alpha_1 I_n$, so $g \in Z(\mathrm{GSp}_n(q, \mathbf{f}_\beta))$.

Let $k = 1$ and $n_1 = 1$. So $\langle e_1 \rangle$ is an S-invariant subspace of V. If $n = 4$, then Theorem C1 is verified by computation. So we can assume that $n > 4$. Thus, $l \geq 2$ and

$$W = \langle f_2 \dots, f_{(n-2)/2}, e_2, \dots, e_{(n-2)/2}, e_1 \rangle$$

is S-invariant. Let $a \in \mathrm{GL}_n(q)$ be such that

$$\begin{aligned}
&(e_1)a = \sum_{i=2}^{(n-2)/2} e_i + e_1 + e; && (f_1)a = f_1; &&\\
&(e_2)a = e_2 + f - f_1; && (f_2)a = f_2 - f_1; && \qquad (5.15)\\
&(e_j)a = e_j; && (f_j)a = f_j - f_1; && j \in \{2,\dots,(n-2)/2\}\\
&(e)a = e + f_2; && (f)a = f - f_1. &&
\end{aligned}$$

It is routine to check that $a \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$. Consider $g \in \tilde{S} \cap S^a$. Since S stabilises the subspaces $\langle e_1 \rangle$ and W, S^a stabilises $\langle (e_1)a \rangle$ and $(W)a$. Therefore,

$$((e_1)a)g = \sum_{i=2}^{(n-2)/2} \delta_i e_i + \alpha_1 e_1 + \lambda_3 f + \lambda_4 e = \eta (e_1)a \tag{5.16}$$

for some $\eta \in \mathbb{F}_q$. Hence $\delta_2 = \ldots = \delta_{(n-2)/2} = \lambda_4 = \alpha_1$ and $\lambda_3 = 0$. In the same way

$$((e_2)a)g = \begin{cases} \delta_2 e_2 + \lambda_1 f + \lambda_2 e - \tau(g)\alpha_1 f_1; \\ \eta_1 (e_1)a + \sum_{i=2}^{(n-2)/2} \eta_i (e_i)a + \sum_{i=2}^{(n-2)/2} \xi_i (f_i)a \end{cases} \tag{5.17}$$

for some $\eta_i, \xi_i \in \mathbb{F}_q$. Since $((e_2)a)g$ does not have e_i for $i > 2$ and f_j for $j > 1$ in the first line of (5.17), $((e_2)a)g = \eta_2(e_2)a$. Therefore, $\tau(g)\alpha_1 = \lambda_1 = \alpha_1$ and $\lambda_2 = 0$, so $g = \alpha_1 I_n \in Z(\mathrm{Sp}_n(q, \mathbf{f}_\beta))$.

Let $k \geq 2$ and $n_1 = 1$. So $\langle e_1 \rangle$ and $W = \langle e_1, e_2, \ldots, e_{(n_2/2+1)} \rangle$ are S-invariant subspaces of V. Let a be as in (5.15). Consider $g \in \tilde{S} \cap S^a$. Since S stabilises the subspaces $\langle e_1 \rangle$ and W, S^a stabilises $\langle (e_1)a \rangle$ and $(W)a$. Therefore, (5.16) holds, so $\delta_2 = \ldots = \delta_{(n-2)/2} = \lambda_4 = \alpha_1$ and $\lambda_3 = 0$. In the same way

$$((e_2)a)g = \begin{cases} \delta_2 e_2 + \lambda_1 f + \lambda_2 e - \alpha_1 f_1; \\ \eta_1 (e_1)a + \sum_{i=2}^{(n_2/2+1)} \eta_i (e_i)a \end{cases} \tag{5.18}$$

for some $\eta_i \in \mathbb{F}_q$. Since $((e_2)a)g$ does not have e_i for $i > 2$ in the first line of (5.18), $((e_2)a)g = \eta_2(e_2)a$. Therefore, $\lambda_1 = \alpha_1$ and $\lambda_2 = 0$, so $g = \alpha_1 I_n \in Z(\mathrm{GSp}_n(q, \mathbf{f}_\beta))$.

Let $k = 0$, so $g \in S$ is a block-diagonal matrix with blocks in $\gamma_i(S) \leq \mathrm{GSp}_{n_i}(q)$. Let $W = \langle f_1, \ldots, f_{n_2/2}, e_1, \ldots, e_{n_2/2} \rangle$. If $n = 4$, then Theorem C1 follows by Lemma 5.5, so let $n \geq 6$. We can assume that $n_2 \geq 4$. Indeed, if $n_i = 2$ for $i \in \{1, \ldots, l\}$, then we can consider $S_1 = \mathrm{diag}[\gamma_2(S), \gamma_3(S)] \leq \mathrm{diag}[\mathrm{GSp}_2(q), \mathrm{GSp}_2(q)]$ as a subgroup in $\mathrm{Sp}_4(q)$. By Lemma 5.5, $b_{S_1}(\mathrm{Sp}_4(q)) \leq 3$. We redefine $\gamma_2(g)$ to be $\mathrm{diag}[\mathrm{GSp}_2(q), \mathrm{GSp}_2(q)]$, so now $n_2 = 4$. Let $m = n_2/2$ and let $a \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$ be defined by (5.13). Arguments similar to the case $(k \geq 1, n_1 \geq 2)$ imply $g \in Z(\mathrm{GSp}_n(q, \mathbf{f}_\beta))$.

Case 2. Let $q = 3$ and $\gamma_s(S)$ is T as defined after Remark 5.4. Without loss of generality, we can assume $s = k+1$. Let β be as in **Step 2** of the proof of Theorem 5.3. Let r be such that the restriction of matrices from S to vectors $\{f_r, f_{r+1}, e_r, e_{r+1}\}$ is $\gamma_s(S)$. We relabel vectors in β as follows:

- f_r, f_{r+1}, e_r and e_{r+1} become f, f_0, e and e_0 respectively;
- if $i < r$, then f_i and e_i remain f_i and e_i respectively;
- if $i > r$, then f_i and e_i become f_{i-2} and e_{i-2} respectively.

Let $y_s, z_s \in \mathrm{Sp}_4(q)$ be such that $\gamma_s(S) \cap \gamma_s(S)^{y_s} \cap \gamma_s(S)^{z_s}$ is as in (1) of Lemma 5.6. For $i \neq s$ define y_i and z_i as in (5.5). Define y and x as in (5.7) and $\tilde{S}$ as in **Step 1** of the proof of Theorem 5.3.

Therefore, $g \in \tilde{S}$ has shape (5.11) with

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \lambda_2 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \in \left\langle 2I_4, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

and $\lambda_i \in \mathbb{F}_3$ for $i \in \{1, 2\}$. Let $W = \langle e_1, \ldots, e_{n_1} \rangle$ if $k > 0$ and

$$W = \langle f_1, \ldots, f_{n_1/2}, e_1, \ldots, e_{n_1/2} \rangle$$

if $k = 0$. Let $m = n_1$ for W totally singular and $m = n_1/2$ for W non-degenerate. Let $a \in \mathrm{GL}_n(q)$ be such that

$$\begin{aligned} (e_1)a &= \sum_{i=m+1}^{(n-4)/2} e_i + e_1 + f + \underline{f_1}; \\ (f_1)a &= f_1; \\ (e_j)a &= \sum_{i=m+1}^{(n-4)/2} e_i + e_j; & j \in \{2, \ldots, m\} \end{aligned}$$

$$\begin{aligned}
&(f_j)a = f_j; && j \in \{2, \ldots, m\} \\
&(e_j)a = e_j; && j \in \{m+1, \ldots, (n-4)/2\} \\
&(f_j)a = f_j - \sum_{i=1}^{m} f_i; && j \in \{m+1, \ldots, (n-4)/2\} \\
&(e)a = e + f_1; && \\
&(f)a = f; && \\
&(e_0)a = e_0; && \\
&(f_0)a = f_0. &&
\end{aligned} \tag{5.19}$$

Here the underlined part is present only if W is totally isotropic. It is routine to check that $a \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$. Since S stabilises the subspace W, S^a stabilises $(W)a$. Therefore,

$$((e_1)a)g = \begin{cases} \sum_{i=m+1}^{(n-4)/2} \delta_i e_i + \alpha_1 e_1 + \lambda_1 f + \lambda_2 e + \underline{\tau(g)\alpha_1 f_1}; \\ \sum_{i=1}^{m} \eta_i (e_i)a + \sum_{i=1}^{m} \xi_i (f_i)a \end{cases} \tag{5.20}$$

for some $\eta_i, \xi_i \in \mathbb{F}_q$. Here all $\xi_i = 0$ if W is totally isotropic. Since $((e_1)a)g$ does not have e_i for $1 < i \leq m$ and f_i for $1 \leq i \leq m$ (for W non-degenerate) in the first line of (5.20), $((e_1)a)g = \eta_1 (e_1)a$, so $\lambda_2 = 0$ and

$$\alpha_1 = \tau(g)\alpha_1 = \lambda_1 = \delta_{m+1} = \ldots = \delta_{(n-4)/2}.$$

Therefore, $g = \alpha_1 I_n$ and $g \in Z(\mathrm{GSp}_n(q, \mathbf{f}_\beta))$.

Case 3. Let $q \in \{3, 5\}$ and $\gamma_s(S) = Q$, so $s \leq k$. If $k + l = 1$, then Theorem C1 follows by (3) of Lemma 5.6. Let β be as in **Step 2** of the proof of Theorem 5.3. Let r be such that the restriction of matrices of S to vectors $\{e_r, e_{r+1}\}$ is $\gamma_s(S)$. We relabel vectors in β as follows:

- f_r, f_{r+1}, e_r and e_{r+1} become f, f_0, e and e_0 respectively;
- if $i < r$, then f_i and e_i remain f_i and e_i respectively;
- if $i > r$, then f_i and e_i become f_{i-2} and e_{i-2} respectively.

Let $\widehat{\beta} = \{f, f_0, e, e_0\}$ and let $\widehat{S}$ be the group consisting of restrictions to $\widehat{\beta}$ of all $g \in S$. Let $\widehat{y}, \widehat{z} \in \mathrm{Sp}_4(q)$ be such that $\widehat{S} \cap \widehat{S}^{\widehat{y}} \cap \widehat{S}^{\widehat{z}}$ is as in (2) of Lemma 5.6 if $q = 3$ and $\widehat{S} \cap \widehat{S}^{\widehat{y}} \cap \widehat{S}^{\widehat{z}} \leq Z(\mathrm{GSp}_4(5))$ if $q = 5$. Let y and z be the $(\widehat{y}, \widehat{\beta})$-replacement and $(\widehat{z}, \widehat{\beta})$-replacement of matrices from (5.7) respectively. Recall that $\tilde{S} = S \cap S^{xy} \cap S^z$ where x is the matrix (5.4). If $q = 5$, then $\tilde{S}$ is a group of diagonal matrices and **Step 2** of the proof of Theorem 5.3 implies the theorem.

If $q = 3$, then $g \in \tilde{S}$ has shape (5.12) with

$$\mathrm{diag}[\tau(g)\Lambda^\dagger, \Lambda] = \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ \lambda_3 & \lambda_4 & 0 & 0 \\ 0 & 0 & \lambda_5 & \lambda_6 \\ 0 & 0 & \lambda_7 & \lambda_8 \end{pmatrix} \in \left\langle \begin{pmatrix} 1&0&0&0 \\ 1&2&0&0 \\ 0&0&1&1 \\ 0&0&0&2 \end{pmatrix}, \begin{pmatrix} 2&2&0&0 \\ 2&1&0&0 \\ 0&0&2&2 \\ 0&0&2&1 \end{pmatrix} \right\rangle$$

and $\lambda_i \in \mathbb{F}_3$ for $i \in \{1, \ldots, 8\}$. Notice that if $\lambda_2 = \lambda_6 = 0$, then $\mathrm{diag}[\tau(g)\Lambda^\dagger, \Lambda]$ is the scalar matrix αI_n with $\alpha \in \mathbb{F}_3^*$, so $\tau(g) = \alpha^2 = 1$.

Assume $s > 1$ and let $W = \langle e_1, \dots, e_m\rangle$, where $m = n_1$. Let $a \in \mathrm{GL}_n(q)$ be such that

$$\begin{array}{lll}
(e_1)a = \sum_{i=m+1}^{(n-4)/2} e_i + e_1 + e + f; & (f_1)a = f_1; & \\
(e_j)a = e_j; & (f_j)a = f_j; & j \in \{2, \dots, m\} \\
(e_j)a = e_j; & (f_j)a = f_j - f_1; & j \in \{m+1, \dots, (n-4)/2\} \\
(e)a = e + f_1; & (f)a = f - f_1; & \\
(e_0)a = e_0; & (f_0)a = f_0. &
\end{array}$$

It is routine to check that $a \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$. Since S stabilises W, S^a stabilises $(W)a$. Therefore,

$$((e_1)a)g = \begin{cases} \sum_{i=m+1}^{(n-4)/2} \delta_i e_i + \alpha_1 e_1 + \lambda_5 e + \lambda_6 e_0 + \lambda_1 f + \lambda_2 f_0; \\ \sum_{i=1}^{m} \eta_i (e_i)a \end{cases} \tag{5.21}$$

for some $\eta_i \in \mathbb{F}_q$. Since $((e_1)a)g$ does not have e_i for $1 < i \leq m$ in the first line of (5.21), $((e_1)a)g = \eta_1(e_1)a$, so $\lambda_2 = \lambda_6 = 0$ and

$$\alpha_1 = \lambda_1 = \lambda_5 = \delta_{m+1} = \ldots = \delta_{(n-4)/2}.$$

Therefore, $g = \alpha_1 I_n$ and $g \in Z(\mathrm{GSp}_n(q, \mathbf{f}_\beta))$.

Assume $s = 1$ and let $W = \langle e, e_0\rangle$. Let $a \in \mathrm{GL}_n(q)$ be such that

$$\begin{array}{lll}
(e)a = e + f; & (f)a = f; & \\
(e_0)a = \sum_{i=1}^{(n-4)/2} e_i + e_0; & (f_0)a = f_0; & \\
(e_j)a = e_j; & (f_j)a = f_j - f_0; & j \in \{1, \dots, (n-4)/2\}.
\end{array}$$

It is routine to check that $a \in \mathrm{Sp}_n(q, \mathbf{f}_\beta)$. Since S stabilises W, S^a stabilises $(W)a$. Therefore,

$$((e)a)g = \begin{cases} \lambda_5 e + \lambda_6 e_0 + \lambda_1 f + \lambda_2 f_0; \\ \eta_1(e)a + \eta_2(e_0)a \end{cases} \tag{5.22}$$

for some $\eta_1, \eta_2 \in \mathbb{F}_q$. Since $((e)a)g$ does not have e_i for $1 \leq i \leq (n-4)/2$ in the first line of (5.22), $((e)a)g = \eta_1(e)a$, so $\lambda_2 = \lambda_6 = 0$ and $\mathrm{diag}[\tau(g)\Lambda^\dagger, \Lambda]$ is the scalar matrix $\lambda_1 I_4$. In the same way

$$((e_0)a)g = \begin{cases} \sum_{i=1}^{(n-4)/2} \delta_i e_i + \lambda_1 e_0; \\ \eta_1(e)a + \eta_2(e_0)a \end{cases} \tag{5.23}$$

for some $\eta_1, \eta_2 \in \mathbb{F}_q$. Since $((e_0)a)g$ does not have e or f in the first line of (5.23), $((e)a)g = \eta_1(e)a$, so $\lambda_1 = \delta_i$ for $i \in 1, \dots, (n-4)/2$ and $g = \lambda_1 I_n \in Z(\mathrm{GSp}_n(q, \mathbf{f}_\beta))$ since $\tau(g) = 1$. □

Theorem C1 now follows by Theorems 5.2, 5.3 and 5.7.

5.2. **Solvable subgroups not contained in** $\Gamma\mathrm{Sp}_n(q)$**.** If $q = 2^f$, then $\mathrm{Sp}_4(q)$ has a graph-field automorphism ψ of order $2f$; see [19, §12.3] for details. If β is a basis of V as in Lemma 2.8, then we can assume that ψ^2 is ϕ_β by [19, Proposition 12.3.3].

Theorem C2. *Let q be even and let $A = \mathrm{Aut}(\mathrm{PSp}_4(q)')$. If S is a maximal solvable subgroup of A, then $b_S(S \cdot \mathrm{Sp}_4(q)') \leq 4$, so $\mathrm{Reg}_S(S \cdot \mathrm{Sp}_n(q)', 5) \geq 5$.*

Proof. For $q = 2$ the statement is verified by computation.

Assume $q = 2^f$ with $f > 1$. Let θ be a generator of $\mathbb{F}_q^*$. By [30, Proposition 2.4.3], $\Delta = \mathrm{Sp}_4(q) \times \langle \theta I_4 \rangle$ where Δ is as defined before Lemma 2.1. Therefore,

$$\mathrm{Aut}(\mathrm{Sp}_4(q)) \cong \mathrm{Sp}_4(q) \rtimes \langle \psi \rangle,$$

and we identify these two groups. Denote $\Gamma := \mathrm{Sp}_4(q) \rtimes \langle \psi^2 \rangle$, so $\Gamma = \mathrm{Sp}_4(q) \rtimes \langle \phi \rangle$.

If S lies in Γ, then the statement follows by Theorem C1.

Assume that S does not lie in Γ, so S is in a maximal subgroup H of A not contained in Γ. For a description of such maximal subgroups see [1, §14] and [7, Table 8.14]. If H is a non-subspace subgroup, then the statement follows by [12, Theorem 1.1]. If H is a subspace subgroup, then H is solvable by [7, Table 8.14], so $S = H$ and $b_S(S \cdot \mathrm{Sp}_4(q)) \leq 3$ by [14, Lemma 5.8]. □

References

[1] M. Aschbacher. On the maximal subgroups of the finite classical groups. *Invent. Math.*, 76(3):469–514, 1984.

[2] R. F. Bailey and P. J. Cameron. Base size, metric dimension and other invariants of groups and graphs. *Bull. Lond. Math. Soc.*, 43(2):209–242, 2011.

[3] A. A. Baykalov. Intersection of conjugate solvable subgroups in symmetric groups. *Algebra Logika*, 56(2):135–149, 2017.

[4] A. A. Baykalov. *Intersection of conjugate solvable subgroups in finite classical groups*. PhD thesis, The University of Auckland, 2021. `https://doi.org/10.48550/arXiv.1703.00124`.

[5] A. A. Baykalov. Base sizes for finite linear groups with solvable stabilisers. *Submitted to Journal of Group Theory*, 2024. https://arxiv.org/abs/2408.08510.

[6] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997.

[7] J. N. Bray, D. F. Holt, and C. M. Roney-Dougal. *The maximal subgroups of the low-dimensional finite classical groups*, volume 407 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2013.

[8] T. C. Burness. Fixed point ratios in actions of finite classical groups, I. *J. Algebra*, 309(1):69–79, 2007.

[9] T. C. Burness. Fixed point ratios in actions of finite classical groups, II. *J. Algebra*, 309(1):80–138, 2007.

[10] T. C. Burness. Fixed point ratios in actions of finite classical groups. III. *J. Algebra*, 314(2):693–748, 2007.

[11] T. C. Burness. Fixed point ratios in actions of finite classical groups. IV. *J. Algebra*, 314(2):749–788, 2007.

[12] T. C. Burness. On base sizes for actions of finite classical groups. *J. Lond. Math. Soc. (2)*, 75(3):545–562, 2007.

[13] T. C. Burness. Simple groups, fixed point ratios and applications. In *Local representation theory and simple groups*, EMS Ser. Lect. Math., pages 267–322. Eur. Math. Soc., Zürich, 2018.

[14] T. C. Burness. Base sizes for primitive groups with soluble stabilisers. *Algebra Number Theory*, 15(7):1755–1807, 2021.

[15] T. C. Burness. On soluble subgroups of sporadic groups. *Israel J. Math.*, 254(1):313–340, 2023.

[16] T. C. Burness, M. W. Liebeck, and A. Shalev. Base sizes for simple groups and a conjecture of Cameron. *Proc. Lond. Math. Soc. (3)*, 98(1):116–162, 2009.

[17] T. C. Burness, E. A. O'Brien, and R. A. Wilson. Base sizes for sporadic simple groups. *Israel J. Math.*, 177:307–333, 2010.

[18] A. A. Buturlakin and M. A. Grechkoseeva. The cyclic structure of maximal tori in finite classical groups. *Algebra Logika*, 46(2):129–156, 2007.

[19] R. W. Carter. *Simple groups of Lie type*. Pure and Applied Mathematics, Vol. 28. John Wiley & Sons, London-New York-Sydney, 1972.

[20] R. W. Carter. *Finite groups of Lie type: Conjugacy classes and complex characters*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1985.

[21] J. Dieudonné. *On the automorphisms of the classical groups*, volume No. 2 of *Memoirs of the American Mathematical Society*. American Mathematical Society, Providence, RI, 1980. With a supplement by Loo Keng Hua [Luo Geng Hua], Reprint of the 1951 original.

[22] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.11.1*, 2021.

[23] D. Gluck and O. Manz. Prime factors of character degrees of solvable groups. *Bull. London Math. Soc.*, 19(5):431–437, 1987.

[24] D. Gorenstein and R. Lyons. The local structure of finite groups of characteristic 2 type. *Mem. Amer. Math. Soc.*, 42(276):vii+731, 1983.

[25] D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 3. Part I. Chapter A*, volume 40 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[26] J. E. Humphreys. *Linear algebraic groups*. Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975.

[27] B. Huppert. *Endliche Gruppen. I*. Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin-New York, 1967.

[28] B. Huppert. Singer-Zyklen in klassischen Gruppen. *Math. Z.*, 117:141–150, 1970.

[29] E. I. Khukhro and V. D. Mazurov, editors. *The Kourovka notebook*. Sobolev Institute of Mathematics. Russian Academy of Sciences. Siberian Branch, Novosibirsk, 2018. Unsolved problems in group theory, Nineteenth edition, March 2019 update.

[30] P. Kleidman and M. Liebeck. *The subgroup structure of the finite classical groups*, volume 129 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1990.

[31] M. Korhonen. *Maximal Solvable Subgroups of Finite Classical Groups*, volume 2346. of *Lecture Notes in Mathematics*. Springer, 2024.

[32] M. W. Liebeck and A. Shalev. Simple groups, permutation groups, and probability. *J. Amer. Math. Soc.*, 12(2):497–520, 1999.

[33] O. Manz and T. R. Wolf. *Representations of solvable groups*, volume 185 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1993.

[34] M. Neunhöffer, F. Noeske, E. A. O'Brien, and R. A. Wilson. Orbit invariants and an application to the Baby Monster. *J. Algebra*, 341:297–305, 2011.

[35] A. Seress. The minimal base size of primitive solvable permutation groups. *J. London Math. Soc. (2)*, 53(2):243–255, 1996.

[36] M. W. Short. *The primitive soluble permutation groups of degree less than* 256, volume 1519 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1992.

[37] D. A. Suprunenko. *Matrix groups*. American Mathematical Society, Providence, R.I., 1976. Translated from the Russian, Translation edited by K. A. Hirsch, Translations of Mathematical Monographs, Vol. 45.

[38] E. P. Vdovin. On the base size of a transitive group with solvable point stabilizer. *J. Algebra Appl.*, 11(1):1250015, 14, 2012. https://arxiv.org/abs/1011.4341.

[39] V. I. Zenkov. Intersections of abelian subgroups in finite groups. *Mat. Zametki*, 56(2):150–152, 1994.

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