On algebraic normalizers of maximal tori in simple groups of Lie type

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Every linear algebraic group (closed subgroup of $GL_n(k)$) is affine. Every affine algebraic group is linear (isomorphic to a closed subgroup of $GL_n(k)$).

Maximal tori and Simple groups

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Example

$\overline{G} = SL_n(k)$ is simple, $\overline{T} = D_n(k) \cap \overline{G}$ is max. torus, $\overline{B}$ is the subgroup of upper triangular matrices, $\overline{B}^-$ – lower triangular matrices.

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$\overline{G} = SL_{l+1}(k)$, $\overline{T} = D_{l+1}(k) \cap \overline{G}$;
$\Phi = A_l = \{\pm(a_i - a_j) | 1 \leqslant i < j \leqslant l+1\}$; $X_{a_i - a_j} = \{E + te_{ij}; t \in k\}$

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$\overline{G} = GL_n(k)$, $\sigma = \phi_q$, $\overline{G}_\sigma = GL_n(q)$.

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$\sigma = \phi_q$, $(\overline{G}_u)_\sigma = O^{p'}((\overline{G}_u)_\sigma) \cong SL_{l+1}(q)$, $(\overline{G}_a)_\sigma \cong PGL_{l+1}(q)$, $O^{p'}((\overline{G}_a)_\sigma) \cong PSL_{l+1}(q)$

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$char(k) = 2$, $\overline{G} = SL_n(k)$, $\sigma = \phi_2$, $\overline{T} = D_n(k) \cap \overline{G}$;
$G = SL_n(2)$,
$\overline{N}$ is the set of monomial matrices, so $N(G, T) \cong \mathrm{Sym}(n)$;
$T = \{I_{n\times n}\}$, so $N_G(T) = G$.

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Lemma

Let $\overline{G}_a = \xi(\overline{G}_u)$. Let $\overline{S} \leqslant \overline{G}_u$ and $\overline{T} = \xi(\overline{S}) \leqslant \overline{G}_a$ be maximal σ-stable tori. Torus $T := \overline{T}_\sigma \cap O^{p'}((\overline{G}_a)_\sigma)$ is nondegenerate in $\overline{G}_a$ if and only if torus $\overline{S}_\sigma$ is nondegenerate in $\overline{G}_u$.

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Carter

If $\overline{G}$ is connected reductive then all maximal tori of $\overline{G}_\sigma$ are nondegenerate provided q is sufficiently large.

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Let $g^\sigma g^{-1} \in \overline{N}$ and $\pi(g^\sigma g^{-1}) = w$. Then $(\overline{T}^g)_\sigma = (\overline{T}_{\sigma w})^g$.

CLASSICAL GROUPS

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On the example of PSL_n^ε

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$\overline{G}_\sigma$-classes of maximal tori $\leftrightarrows$ conjugacy classes of elements of $Sym(n)$.
The standard representative of a class is

$$(1, \dots, n_1)(n_1 + 1, \dots, n_1 + n_2) \dots (n - n_m + 1, \dots, n)$$

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Buturlakin and Grechkoseeva

Let w be the standard representative of type $(n_1)(n_2)\dots(n_m)$, U be the subgroup of $GL_n(k)$ of all block-diagonal matrices $bd(D_1, \dots, D_m)$ such that $D_i = \mathrm{diag}(\lambda_i, \lambda_i^{\varepsilon q}, \dots, \lambda_i^{(\varepsilon q)^{n_i-1}})$, $\lambda_i^{(\varepsilon q)^{n_i}-1} = 1$, for all $i \in \{1, 2, \dots, m\}$. Then $\overline{T}_{\sigma w} = U \cap \overline{G}$.

(B.) Theorem 1

Let $\overline{G}$ be $SL_n(k)$ and $\sigma \in \{\phi_q, \phi_q\tau\}$, so $\overline{G}_\sigma = SL_n^\epsilon(q)$. Let $\overline{T}$ be the subgroup of all diagonal matrices in $\overline{G}$. If $\overline{T}^g$ is a maximal σ-stable torus, then $S := (\overline{T}^g)_\sigma$ is nondegenerate always, except the case $\overline{G}_\sigma = SL_n(2)$ and $w := \pi(g^\sigma g^{-1})$ is of type $(n_1)(n_2)\dots(n_m)$, where $n_1 = n_2 = 1$.

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(B.) Theorem 2

Let $\overline{G}$ be $PSL_n(k) = PGL_n(k)$ and $\sigma \in \{\phi_q, \phi_q\tau\}$, such that $G = PSL_n^\epsilon(q)$ is a finite simple group. Let $\overline{T} \leqslant \overline{G}$ be the image of the subgroup of all diagonal matrices of $GL_n(k)$, so $\overline{T}$ is a maximal σ-stable torus. If $\overline{T}^g$ is a maximal σ-stable torus of $\overline{G}$, then $S := (\overline{T}^g)_\sigma \cap G$ is nondegenerate always, except the case $G = PSL_n(2)$ and $w := \pi(g^\sigma g^{-1})$ is of type $(n_1)(n_2)\dots(n_m)$, where $n_1 = n_2 = 1$.

EXCEPTIONAL GROUPS

(B.) Theorem 3

$N(G, T) = N_G(T)$ for all tori of groups

$$
\begin{array}{cc}
{}^2G_2(3^{2n+1}); & {}^2B_2(2^{2n+1}); \\
{}^2F_4(2^{2n+1}); & {}^3D_4(q^3); \\
G_2(q), q > 3; & F_4(q), q > 3; \\
{}^2E_6(q^2), q > 3; & E_6(q), q > 3; \\
E_7(q), q > 3; & E_8(q), q > 5;
\end{array}
$$

Idea of the proof

$N(G, T) = N_G(T)$ if there are no root subgroups in $\overline{R} := C_{\overline{G}}(T)^0$.

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If for every reductive subgroup of maximal rank R of G, $|T|$ does not divide $|Z(R)|$, then there are no $\overline{T}$-root subgroups in $C_{\overline{G}}(T)^0$.

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Orders of T are known. If q is not "bad" then $\overline{R}$ is the connected centralizer of a semisimple element and corresponding orders of $|Z(R)|$ are also known.