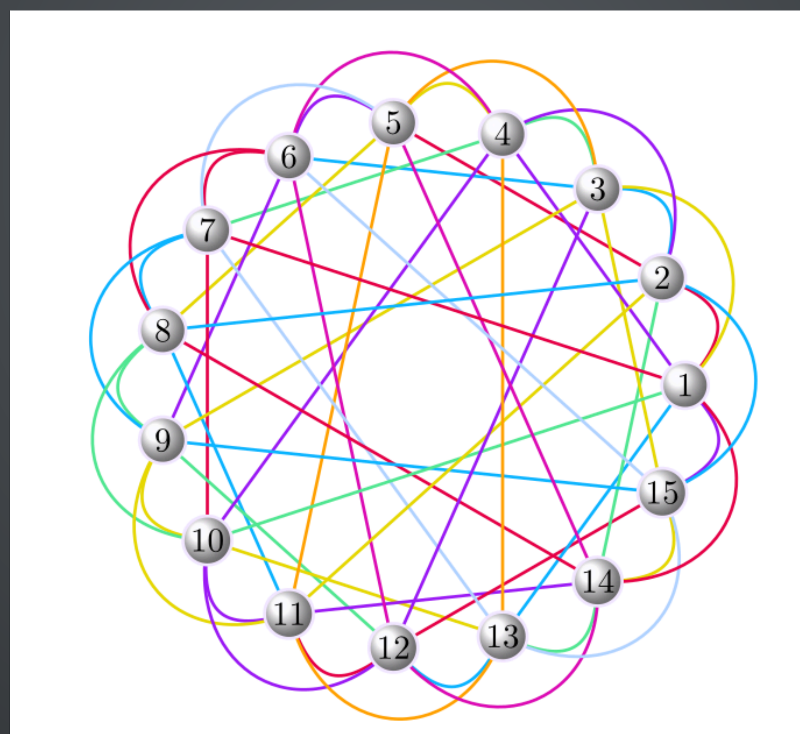
IMPRIMITIVE PARTIAL LINEAR SPACES AND GROUPS OF RANK 3

ANTON A. BAYKALOV

Groups in Galway 2025



JOINT WORK WITH

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A partial linear space $S = (\mathcal{P}, \mathcal{L})$:

- each line is incident with a constant number $k \geq 2$ of points
- each point is incident with a constant number r of lines
- each pair of points is incident with at most one line

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Goal

Find/classify/describe highly symmetrical partial linear spaces

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Example

- if $k = 2$ then S is a **graph**
- if each pair of points is collinear the S is a **linear space**

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PLS is *proper* if $k > 2$ at least one pair of points is not collinear.

HIGHLY SYMMETRICAL LINEAR SPACES

Kantor [1] classified the 2-transitive linear spaces: automorphism group acts transitively on ordered pairs of distinct points.

Examples

- 🍀 $\mathrm{PG}_n(q)$
- 🍀 $\mathrm{AG}_n(q)$
- 🍀 Ree unitals

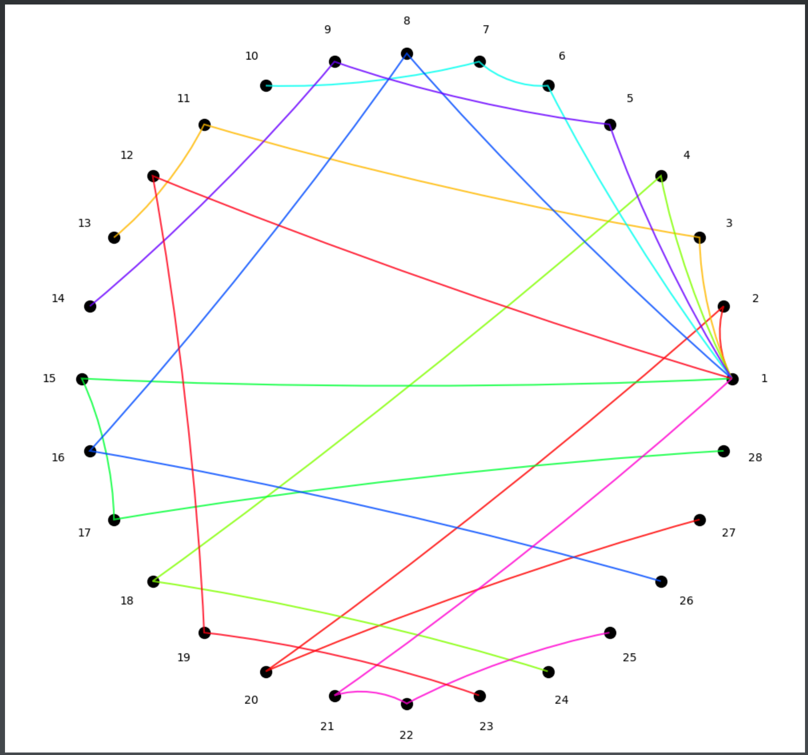
References

[1] Kantor, W. M. Homogeneous designs and geometric lattices. J. Combin. Theory, Ser. A 38 (1985), 66–74.

HIGHLY SYMMETRICAL LINEAR SPACES

Ree unitals

- Let $G = \mathrm{Ree}(q) = {}^2G_2(q), \quad q = 3^{2m+1}$
- G acts by conjugation on the set $\mathcal{P}$ of Sylow 3-subgroups ($|\mathcal{P}| = q^3 + 1$)
- $G_{A,B}$ is cyclic of order $q - 1$ and contains unique involution σ for ($A, B \in \mathcal{P}$)
- σ has $q + 1$ fixed points
- $\mathcal{L}$ consists of fixed point sets for all such σ's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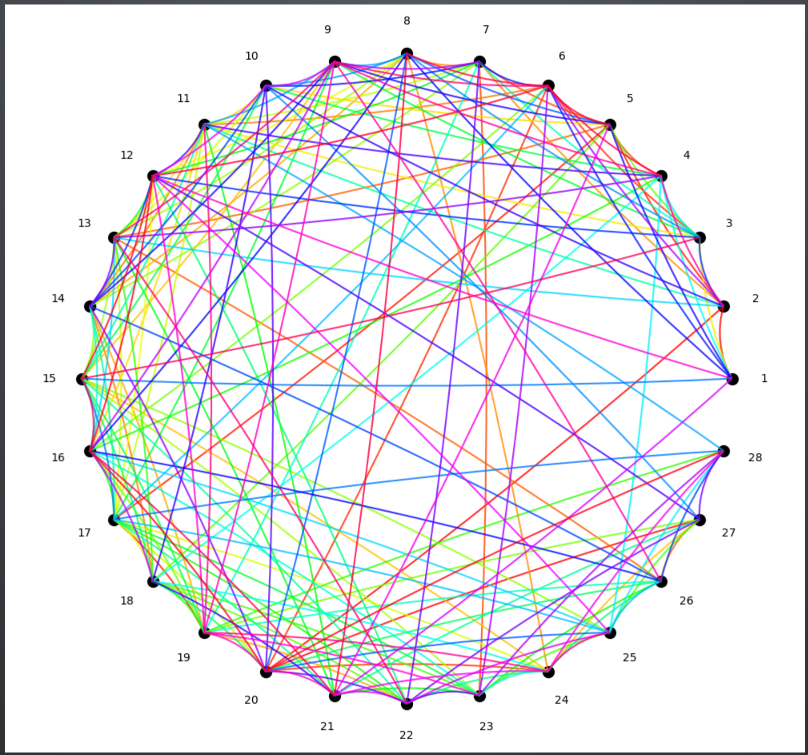
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MORE GENERALLY

Flag-transitive linear spaces S: automorphism group G acts transitively on point-line incident pairs.

Classification in [2]. In particular, G acts primitively on points of S and either G is almost simple or $G \leq \mathrm{A\Gamma L}_n(q) = \mathbb{F}_q^n \rtimes \mathrm{GL}_n(q)$.

Examples

🍀 $\mathrm{PG}_n(q)$ 🍀 Witt spaces 🍀 Hermitian unitals 🍀 Ree unitals

References

[2] Buekenhout, F., Delandtsheer, A., Doyen, J., Kleidman, P. B., Liebeck, M. W., and Saxl, J. Linear spaces with flag-transitive automorphism groups. Geom. Dedic. 36 (1990), 89–94.

GENERALISING TO PARTIAL LINEAR SPACES

Consider those partial linear spaces for which automorphism group G acts transitively on

1. Ordered pairs of distinct collinear points
2. Ordered pairs of distinct non-collinear points

Such PLS are flag-transitive, and when they have non-empty line sets and are not linear spaces, they are precisely:

Partial linear spaces S for which $\mathrm{Aut}(S)$ is transitive of rank 3 on points.

Rank

For $G \leq \mathrm{Sym}(\Omega)$, rank is the number of orbits of G on $\Omega \times \Omega$. If G is transitive, this is equal to the number of orbits of a point stabiliser.

Rank 3 Graphs <-> Rank 3 Groups

Rank 3 group action gives us collinearity relations for vertices which is enough to reconstruct the graph (up to complement).

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Let Ω be $\mathbb{F}_q^n \setminus \{0\}, n \geq 2, q \geq 3$

- $\mathrm{AG}^*(n, q) = (\Omega, \mathcal{L})$ where $\mathcal{L} = \{L_{u,v} \mid u, v \in \mathbb{F}_q^n, \dim(\langle u, v \rangle) = 2\}$, and $L_{u,v} = \{\lambda u + (1 - \lambda) v \mid \lambda \in \mathbb{F}_q\}$.
- $\Delta(n, q) = (\Omega, \mathcal{L})$ where $\mathcal{L} = \{L_{u,v} \mid u, v \in \mathbb{F}_q^n, \dim(\langle u, v \rangle) = 2\}$ with $L_{u,v} = \{u, v, -(u + v)\}$.

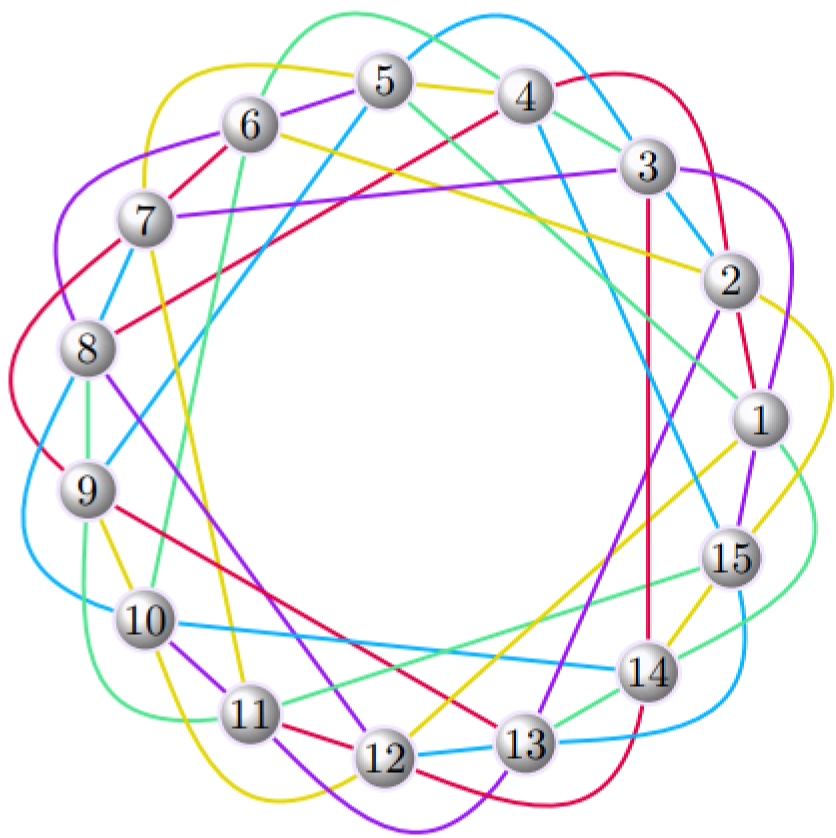

FIGURE 2.1. AG*(2, 4)

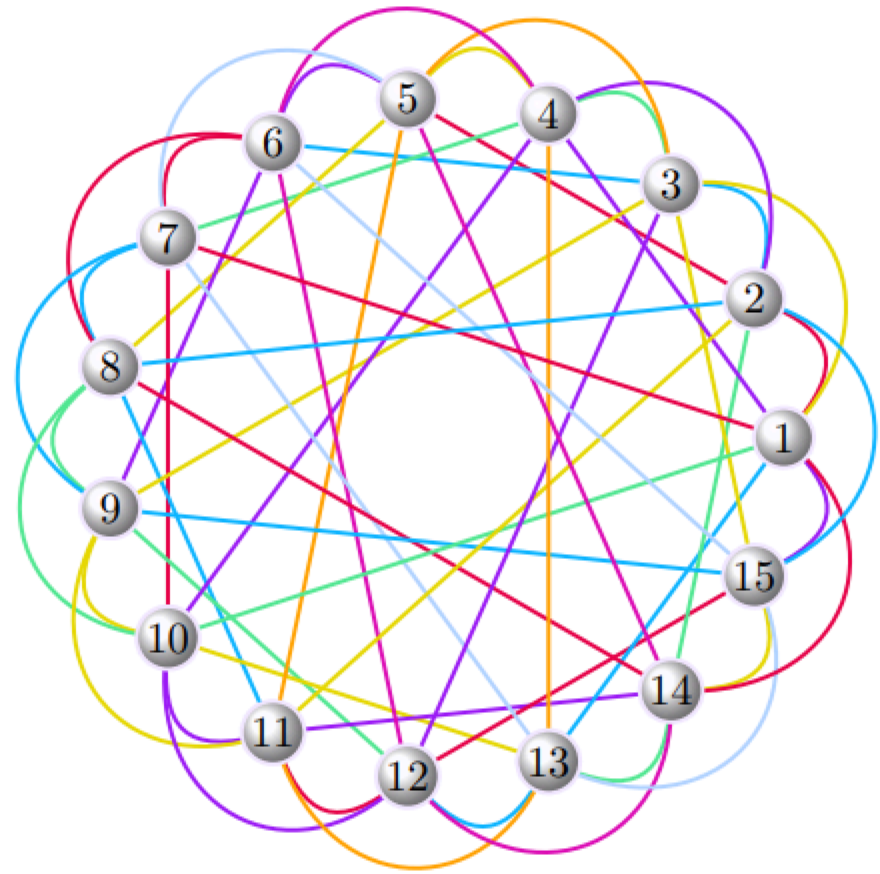

FIGURE 2.2. $\Delta(2, 4)$

Theorem 1 (Devillers, 2005)

Let G be a rank 3 permutation group on a set $\mathcal{P}$. Suppose that G is an automorphism group of a proper partial linear space $S = (\mathcal{P}, \mathcal{L})$. Then one of the following holds:

1. G is imprimitive on $\mathcal{P}$ and S is a disjoint union of same-sized lines, namely the blocks of imprimitivity of G, or
2. for any point $\alpha \in \mathcal{P}$ and any line $L \in \mathcal{L}$ through α, the set $L \backslash \{\alpha\}$ is a block of imprimitivity for G_α, and the stabilizer G_L is transitive on L.

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Theorem 2 (Devillers, 2005)

Let G be a rank 3 permutation group on a set $\mathcal{P}$ and let α be a point in $\mathcal{P}$. If, in the action of G_α on any of its orbits, we choose a block B of imprimitivity such that the stabilizer of $B \cup \{\alpha\}$ in G is transitive on the points of $B \cup \{\alpha\}$, then the pair $(\mathcal{P}, \mathcal{L})$, where $\mathcal{L} = (B \cup \{\alpha\})^G$, forms a proper partial linear space.

RANK 3 PERMUTATION GROUPS

$G \leq \mathrm{Sym}(\Omega)$ (transitive)

- Primitive: G does not preserve a block system on Ω (classic)
- Quasiprimitive: every normal subgroup of G is transitive [3]
- Innately transitive: $\exists$ transitive minimal normal subgroup of G [4]
- Semiprimitive*: each normal subgroup of G is either transitive or semiregular [5]

References

[3] A. Devillers, M. Giudici, C. H. Li, G. Pearce, and C. E. Praeger. (2011)

[4] A. A. Baykalov, A. Devillers, and C. E. Praeger. (2023)

[5] Huang, H. Y., Li, C. H., & Zhu, Y. Z. (2025).

PRIMITIVE RANK 3 PARTIAL LINEAR SPACES

Primitive rank 3 groups

(a) Grid type: $T \times T \trianglelefteq G \leq T \wr \mathrm{Sym}_2$ on n^2 points with T almost simple 2-transitive on n points

(b) Almost simple.

(c) Affine.

The primitive rank 3 partial linear spaces are essentially classified

- almost simple type, grid type – classified by Devillers (2005, 2008)
- affine type – a satisfactory classification except for a few ‘hopeless’ cases (Bamberg, Devillers, Fawcett, and Praeger – 2021)

PRIMITIVE RANK 3 PARTIAL LINEAR SPACES

There are a lot of interesting primitive rank 3 partial linear spaces:

- classical symplectic, hermitian and orthogonal polar spaces
- Fischer spaces
- Buildings of type E_6
- so on...

IMPRIMITIVE RANK 3 PERMUTATION GROUPS

Let $G \leq \mathrm{Sym}(\Omega)$ be imprimitive of rank 3.

Lemma

There exists unique system of imprimitivity Σ and both G^{Σ} and G_{σ}^{σ} are 2-transitive.

- If G is innately transitive, then G^{Σ} is almost simple .
- If G is quasiprimitive iff G acts on Σ faithfully, so $G^{\Sigma} \cong G$.

OBSERVATIONS ON INNATELY TRANSITIVE GROUPS

$G \leq \mathrm{Sym}(\Omega)$ is properly innately transitive

- G has a plinth M (transitive minimal normal subgroup)
- $C := C_G(M)$ sat. $1 \neq C \lhd G$ and C is semiregular and intransitive
- M is nonabelian as otherwise $M \subseteq C$ forcing C to be transitive
- M is the unique plinth of G, so
- $M = T^k$ for some nonabelian finite simple group T and integer $k \geq 1$
- $C \times M \leq G$

[6] J. Bamberg and C. E. Praeger, "Finite permutation groups with a transitive minimal normal subgroup", Proc. London Math. Soc. .3/ 89:1 (2004), 71–103.

OBSERVATIONS ON INNATELY TRANSITIVE GROUPS

$G \leq \mathrm{Sym}(\Omega)$ is properly innately transitive

- The set Σ of C-orbits on Ω is a G-invariant partition of Ω. Let $\sigma \in \Sigma$ and $\alpha \in \sigma$.
- $G^\Sigma = G/C$ is quasiprimitive with unique plinth $M^\Sigma = MC/C \cong M$;
- $M_\alpha \lhd M_\sigma$ and the induced group $(M_\sigma)^\sigma = M_\sigma/M_\alpha \cong C$ and is regular on σ.

Definition

$\varphi : \mathcal{PIT} \to \mathcal{Q}$ given by $\varphi(G^\Omega) = G^\Sigma$

$\widehat{\varphi} : G^\Omega \to (G^\Sigma, R)$ with $R = (M_\alpha)^\Sigma$

We show that for each group $G \in \mathcal{PIT}_3$ with G acting on a set Ω, the image $\widehat{\varphi}(G^\Omega) = (G^\Sigma, R)$ is such that

- G^Σ is an almost simple 2-transitive group with nonabelian simple socle $M^\Sigma \cong M$
- $1 \neq R = (M_\alpha)^\Sigma \lhd (M^\Sigma)_\sigma$ for some $\sigma \in \Sigma$, and R is $(G^\Sigma)_\sigma$-invariant
- $(M^\Sigma)_\sigma / R$ is elementary abelian and nontrivial (and isomorphic to C)
- $(G^\Sigma)_\sigma$ acts transitively by conjugation on the nontrivial elements of $(M^\Sigma)_\sigma / R$

We call (G^Σ, R) having these properties special pairs

Hypothesys 1.

Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive imprimitive permutation group of rank 3 that is either innately transitive or semiprimitive such that G^Σ is almost simple, where Σ is the unique nontrivial system of imprimitivity. Let M be the socle of G^Σ and let r be the size of a block.

M	$\lvert\Sigma\rvert$	r	G	Conditions on G	type
$\mathrm{PSL}_n(q)$	$\frac{q^n-1}{q-1}$	prime such that $r \mid (q-1)$, $o_r(p) = r-1$ and $(n,r) \neq (2,2)$	$\langle \omega^r I\rangle\,\mathrm{SL}_n(q)/\langle \omega^r I\rangle \leqslant G \leqslant \Gamma\mathrm{L}_n(q)/\langle \omega^r I\rangle$	$\lvert G^\Sigma/(G^\Sigma \cap \mathrm{PGL}_n(q))\rvert = a/j$ with $(j, r-1) = 1$	qp/it/sp
$\mathrm{PSL}_2(q)$	$q+1$	2	$\langle \omega^2 I\rangle\,\mathrm{SL}_2(q)/\langle \omega^2 I\rangle \leqslant G \leqslant \Gamma\mathrm{L}_2(q)/\langle \omega^2 I\rangle$	$q \geqslant 5$, q is odd, $G^\Sigma \nleqslant \mathrm{P\Sigma L}_2(q)$	qp/it/sp
$\mathrm{PSU}_3(q)$	q^3+1	odd prime such that $r \mid q-1$, $o_r(p) = r-1$	$\langle \omega I\rangle\,\mathrm{SU}_3(q)/\langle \omega^r I\rangle \leqslant G \leqslant \Gamma\mathrm{U}_3(q)/\langle \omega^r I\rangle$	$\lvert G^\Sigma/(G^\Sigma \cap \mathrm{PGU}_3(q))\rvert = 2a/j$ with $(j, r-1) = 1$	it
$\mathrm{PSL}_3(2)$	7	2	$\mathrm{PSL}_3(2)$, $C_2 \times \mathrm{PSL}_3(2)$		qp, it
M_{11}	11	2	M_{11}, $C_2 \times \mathrm{M}_{11}$		qp, it
$\mathrm{PSL}_3(4)$	21	6	$\mathrm{PGL}_3(4)$, $\mathrm{P\Gamma L}_3(4)$		qp, qp
$\mathrm{PSL}_3(5)$	31	5	$\mathrm{PSL}_3(5)$		qp
$\mathrm{PSL}_5(2)$	31	8	$\mathrm{PSL}_5(2)$		qp
$\mathrm{PSL}_3(8)$	73	28	$\mathrm{P\Gamma L}_3(8)$		qp
$\mathrm{PSL}_3(3)$	13	3	$\mathrm{PSL}_3(3)$		qp
$\mathrm{Alt}(6)$	6	3	$3.\mathrm{Sym}(6)$	$G_\alpha = \mathrm{Alt}(5)$	sp
M_{12}	12	2	$2.\mathrm{M}_{12}$	$G_\alpha = \mathrm{M}_{11}$	sp

TABLE 1.1. G satisfying Hypothesis 1.1, $q = p^a$ with p prime

Theorem 2 (Devillers, 2005)

Let G be a rank 3 permutation group on a set $\mathcal{P}$ and let α be a point in $\mathcal{P}$. If, in the action of G_α on any of its orbits, we choose a block B of imprimitivity such that the stabilizer of $B \cup \{\alpha\}$ in G is transitive on the points of $B \cup \{\alpha\}$, then the pair $\mathcal{D} = (\mathcal{P}, \mathcal{L})$, where $\mathcal{L} = (B \cup \{\alpha\})^G$, forms a proper partial linear space.

Note

In our case, G_α has orbits $\{\alpha\}$, $\sigma \backslash \{\alpha\}$ and $\Omega \backslash \sigma$.

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Note

In our case, G_α has orbits $\{\alpha\}, \sigma \backslash \{\alpha\}$ and $\Omega \backslash \sigma$.

If $B \subseteq \sigma \backslash \{\alpha\}$, then $\mathcal{D}$ is disconnected.

ACTION OF $\Gamma L_n(q)/\langle \omega^r I \rangle$

Let $q = p^a, a \geq 1, q \geq 3$, and $\langle \omega \rangle = \mathbb{F}_q^*$.

Let $n \geq 2$ with $(n, q) \neq (2, 3)$ and let $r > 1$ be an integer dividing $q - 1$.

Let $V = \mathbb{F}_q^n$ denote the space of n-dimensional row vectors, and $\binom{V}{1}$ the set of 1-subspaces of V.

Define $\Omega := \{\langle \omega^r \rangle u \mid u \in V^*\}$

Then the natural induced action of $\Gamma L_n(q)$ on Ω given by

$g : \langle \omega^r \rangle v \mapsto \langle \omega^r \rangle (v) g$, for $g \in \Gamma L_n(q)$ and $v \in V^*$

yields a permutation group $\overline{G} = \Gamma L_n(q)/Y$ where $Y = \langle \omega^r I \rangle < \Gamma L_n(q)$.

The group $\overline{G}$ stabilises the partition

$\Sigma = \{\sigma(U) \mid U \in \binom{V}{1}\}$

where $\sigma(U) = \{\langle \omega^r \rangle \omega^i u \mid 0 \leq i < r\}$ for $U = \langle u \rangle \in \binom{V}{1}$.

Suppose that G is a group satisfying $Y\,SL_n(q)/Y \leq G \leq \overline{G}$. Then the following hold.

- G is semiprimitive on Ω;
- G is innately transitive on Ω if and only if r divides $(q-1)/(n,q-1)$; and in this case $Y\,SL_n(q)/Y \cong PSL_n(q)$ is the unique plinth of G;
- G is quasiprimitive on Ω if and only if r divides $(q-1)/(n,q-1)$ and $G \cap (Z/Y) = 1$;
- G has rank 3 on Ω if and only if one of the following holds:

1. $(n,r) \neq (2,2)$, r is a primitive prime divisor of $p^{r-1}-1$ and $(r-1, \frac{a}{|G:G\cap GL_n(q)/Y|}) = 1$;
2. $(n,r) = (2,2)$ and $G \not\leq Y\,\Sigma\mathrm{L}_2(q)/Y$.

Let Ω be $\mathbb{F}_q^n \backslash \{0\}, n \geq 2, q \geq 3$

- $\mathrm{AG}^*(n,q) = (\Omega, \mathcal{L})$; $\quad \mathcal{L} = \{L_{u,v} \mid u, v \in \mathbb{F}_q^n, \dim(\langle u, v\rangle) = 2\}$, and $L_{u,v} = \{\lambda u + (1-\lambda)v \mid \lambda \in \mathbb{F}_q\}$.
- $\Delta(n,q) = (\Omega, \mathcal{L})$; $\quad \mathcal{L} = \{L_{u,v} \mid u, v \in \mathbb{F}_q^n, \dim(\langle u, v\rangle) = 2\}$ with $L_{u,v} = \{u, v, -(u+v)\}$.

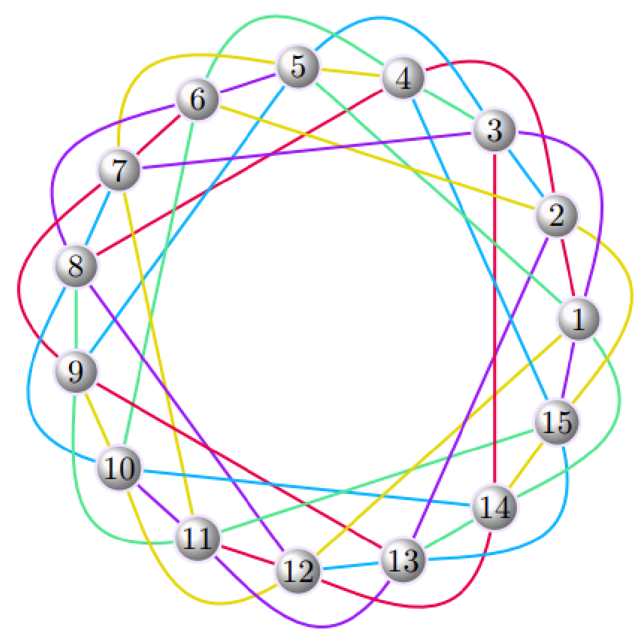


FIGURE 2.1. $\mathrm{AG}^*(2,4)$

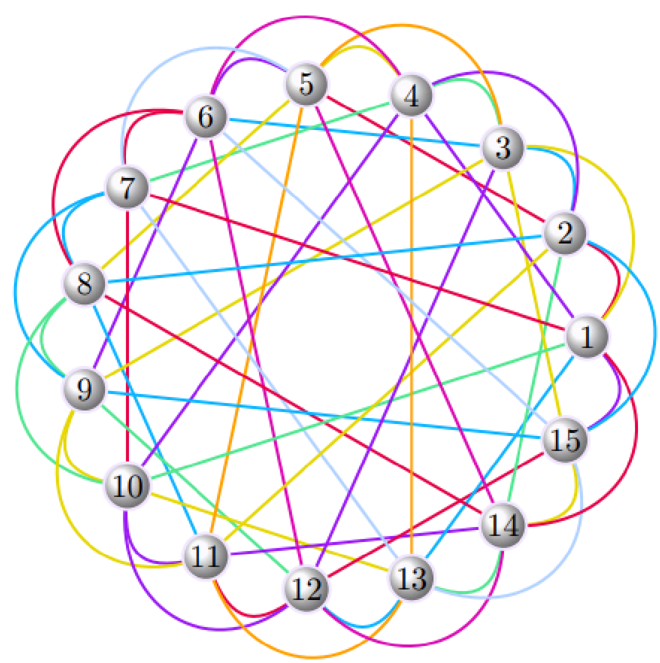


FIGURE 2.2. $\Delta(2,4)$

$\mathcal{D}$	Reference	r	Possible G	Type of G
$\mathrm{AG}^*(n,4)$, $n \geqslant 2$	Definition 2.2	3	$\langle \mathrm{SL}_n(4), \mathrm{diag}(1,\dots,1,\omega)\phi\rangle$, $\mathrm{SL}_n(4) \rtimes \langle\phi\rangle, \Gamma\mathrm{L}_n(4)$	qp/it/sp
$\Delta(n,3)$, $n \geqslant 3$	Definition 2.3	2	$\mathrm{SL}_n(3), \mathrm{GL}_n(3)$	qp/it/sp
$\Delta(n,4)$, $n \geqslant 2$	Definition 2.3	3	$\langle \mathrm{SL}_n(4), \mathrm{diag}(1,\dots,1,\omega)\phi\rangle$, $\mathrm{SL}_n(4) \rtimes \langle\phi\rangle, \Gamma\mathrm{L}_n(4)$	qp/it/sp
$\mathrm{LSub}(n,16,4,5)$, $n \geqslant 2$	Definition 2.9	5	G of rank 3 such that $G \geqslant \langle\omega^5 I\rangle\,\mathrm{SL}_n(16)/\langle\omega^5 I\rangle$ and $G \leqslant \Gamma\mathrm{L}_n(16)/\langle\omega^5 I\rangle$	qp/it/sp
$\mathrm{LSub}(2,81,9,5)$	Definition 2.9	5	$(Z\,\mathrm{SL}_2(3^4) \rtimes \langle\phi\rangle)/\langle\omega^5 I\rangle$	it
$\mathrm{LSub}(2,25,5,3)$	Definition 2.9	3	$(Z\,\mathrm{SL}_2(5^2) \rtimes \langle\phi\rangle)/\langle\omega^3 I\rangle$	it
$\mathrm{DLSub}(9,3,2,1)$	Definition 2.13	2	$\langle\langle\omega^2 I\rangle\,\mathrm{SL}_2(3^2), \phi\,\mathrm{diag}(1,\omega)\rangle/\langle\omega^2 I\rangle$	qp
$\mathrm{USub}(4,2,3)$	Definition 3.6	3	$\Gamma\mathrm{U}_3(4)/\langle\omega^3 I\rangle$	it
$\mathrm{USub}(16,4,5)$	Definition 3.6	5	$\Gamma\mathrm{U}_3(16)/\langle\omega^5 I\rangle$	it
$\mathrm{AGU}^*(4)$	Definition 3.12	3	$\Gamma\mathrm{U}_3(4)/\langle\omega^3 I\rangle$	it

TABLE 1.2. Partial linear spaces arising from linear and unitary groups

M	$\lvert\Sigma\rvert$	r	$\mathcal{D}$	n. of lines	size of lines	Possible G	type
$\mathrm{PSL}_3(3)$	13	3	$\mathcal{D}_1$	234	3	$\mathrm{PSL}_3(3)$	qp
			$\mathcal{D}_2$	117	4	$\mathrm{PSL}_3(3)$	qp
	7	2	$\mathcal{D}_3$	14	4	$\mathrm{PSL}_3(2)$, $C_2 \times \mathrm{PSL}_3(2)$	qp, it
			$\mathcal{D}_4$	28	3	$\mathrm{PSL}_3(2)$	qp
$\mathrm{PSL}_3(8)$	73	28	$\mathcal{D}_5$	686784	3	$\mathrm{P\Gamma L}_3(8)$	qp
			$\mathcal{D}_6$	98112	7	$\mathrm{P\Gamma L}_3(8)$	qp
$\mathrm{PSL}_5(2)$	31	8	$\mathcal{D}_7$	248	16	$\mathrm{PSL}_5(2)$	qp
$\mathrm{PSL}_3(5)$	31	5	$\mathcal{D}_8$	775	6	$\mathrm{PSL}_3(5)$	qp
			$\mathcal{D}_9$	3875	3	$\mathrm{PSL}_3(5)$	qp
$\mathrm{PGL}_3(4)$	21	6	$\mathcal{D}_{10}$	2520	3	$\mathrm{PGL}_3(4)$	qp

$\mathrm{LSub}(n, q, q_0, r) = (\Omega, \mathcal{L})$

Let $n \geq 2$ and fix a basis $\{e_1, e_2, \ldots, e_n\}$ of V. Assume that $q = q_0^f$ for a prime power q_0 and an integer $f > 1$, and also that $\frac{q-1}{q_0-1} = rk$ for some integer k.

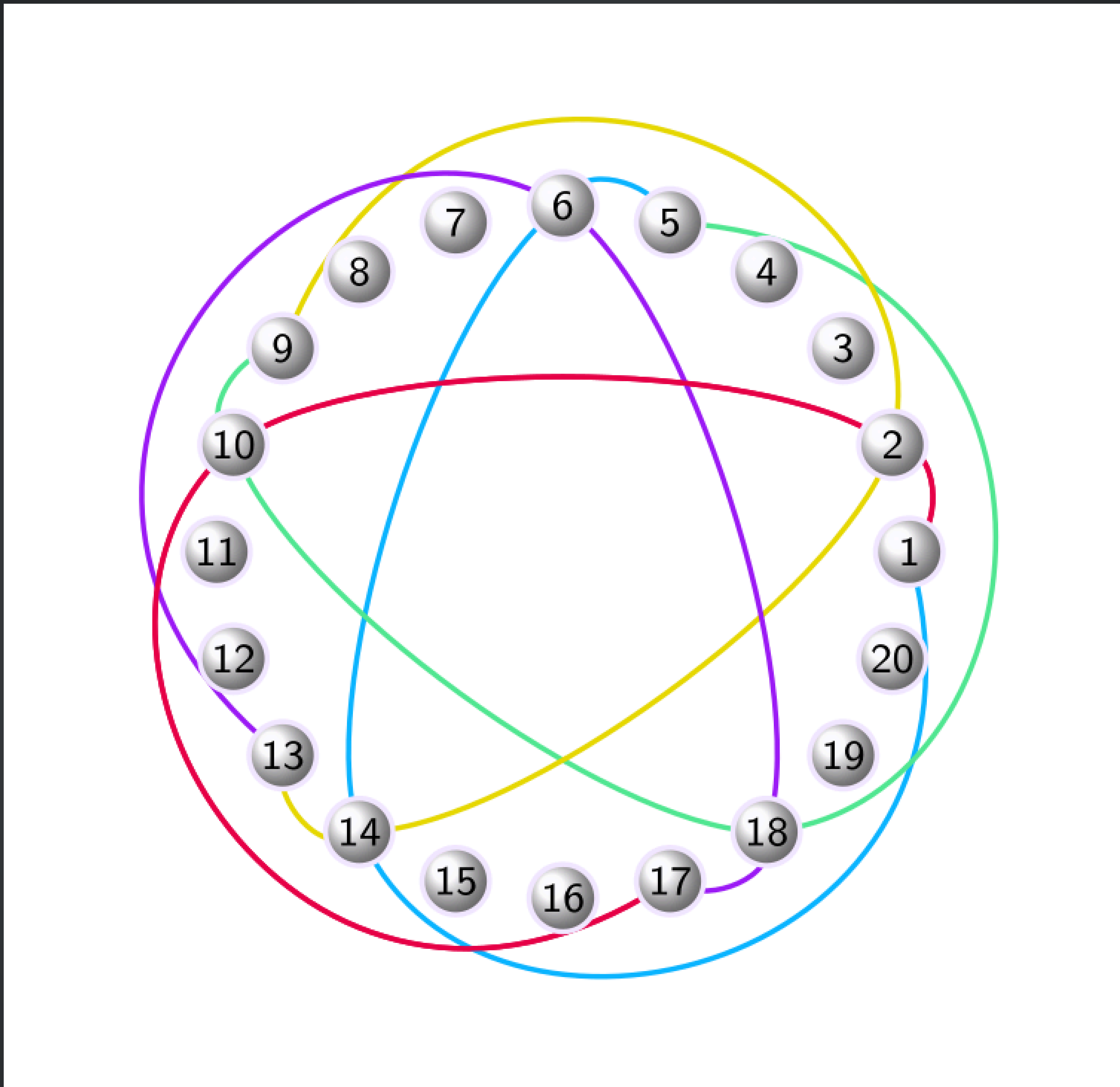
Let t be the least positive integer such that $\langle \omega^t \rangle \cap \langle \omega^r \rangle = \langle \omega^{kr} \rangle$. Define the following sets:

- $L_{u,v} = \langle \omega^r \rangle \{\lambda_1 u + \lambda_2 v \mid \lambda_1, \lambda_2 \in \mathbb{F}_{q_0}, (\lambda_1, \lambda_2) \neq (0,0)\}$ for linearly independent $u, v \in V$
- $\mathcal{L} = \{L_{u,v} \mid (u,v) = (e_1, e_2)^g, g \in GL_n(q), \det(g) \in \langle \omega^t \rangle\}$

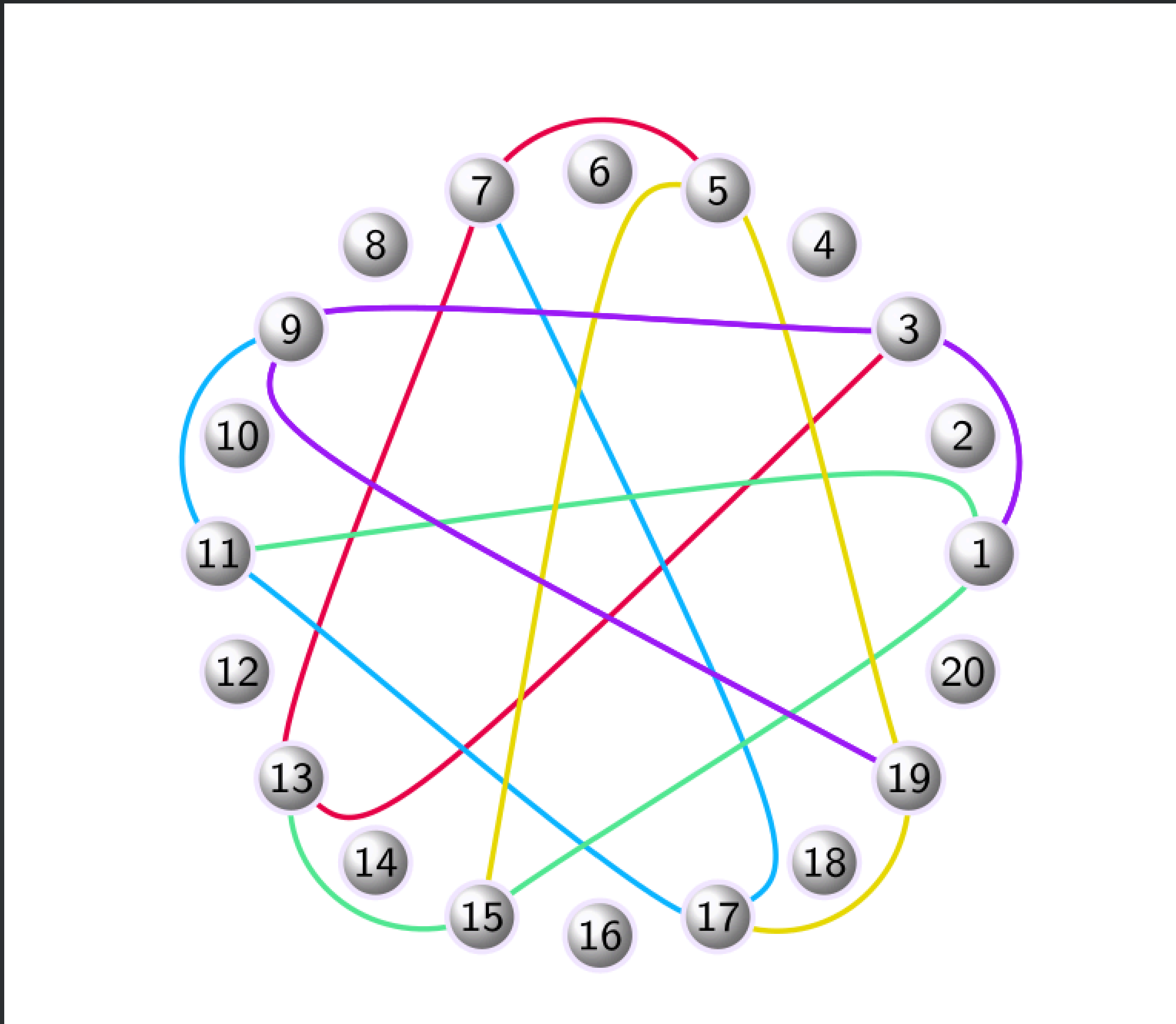
$\mathrm{DLSub}(q, q_0, r, j) = (\Omega, \mathcal{L} \cup \omega^j \mathcal{L})$

For $0 < j < t$:

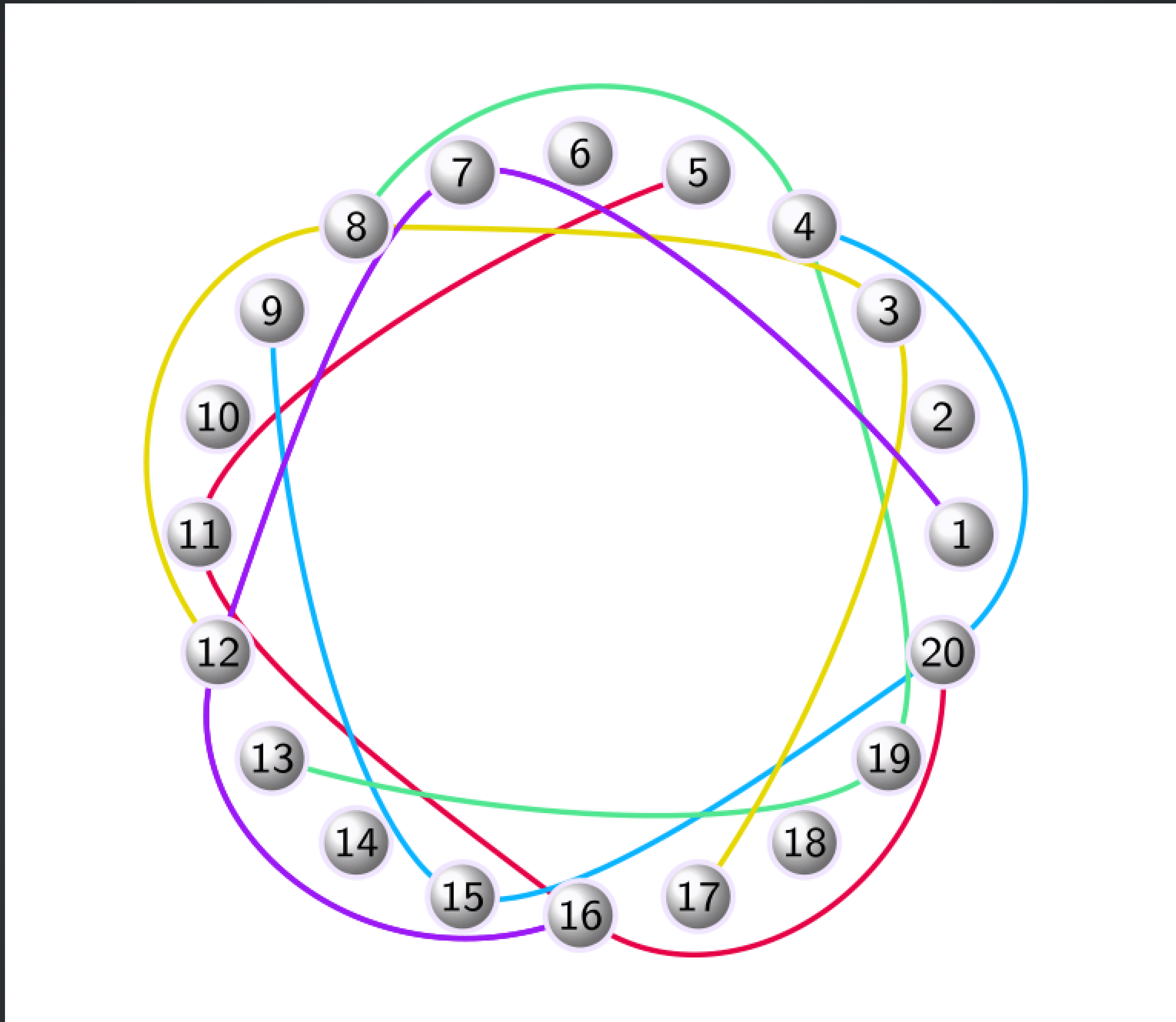
$\omega^j \mathcal{L} = \{L_{u,v} \mid (u,v) = (\omega^j e_1, e_2)^g, g \in GL_2(q), \det(g) \in \langle \omega^t \rangle\}$.



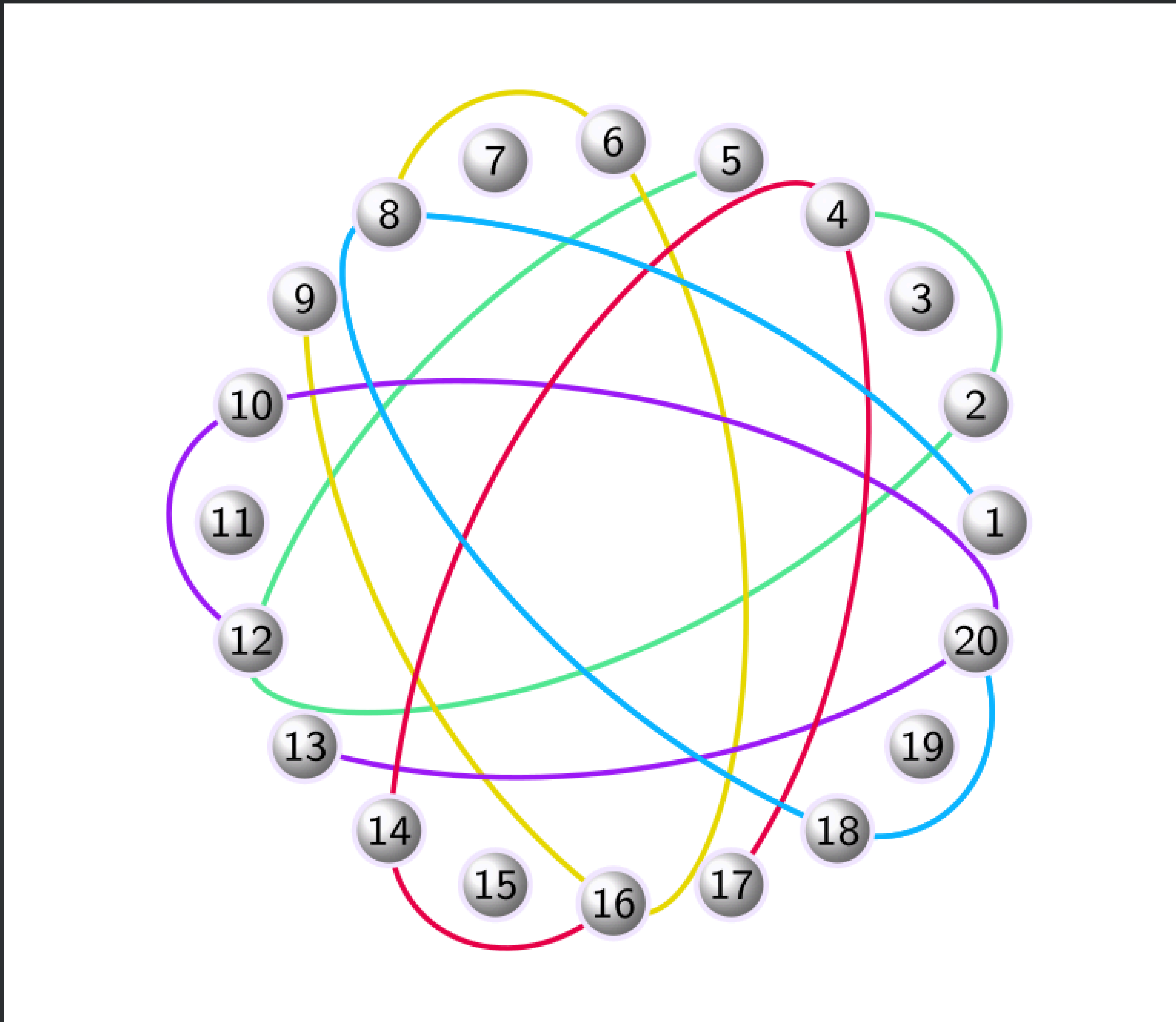
$$\mathrm{DLSub}(9,3,2,1),(1,5,9,13,17)\ldots(4,8,12,16,20)\rightarrow[1,2,10,17]$$



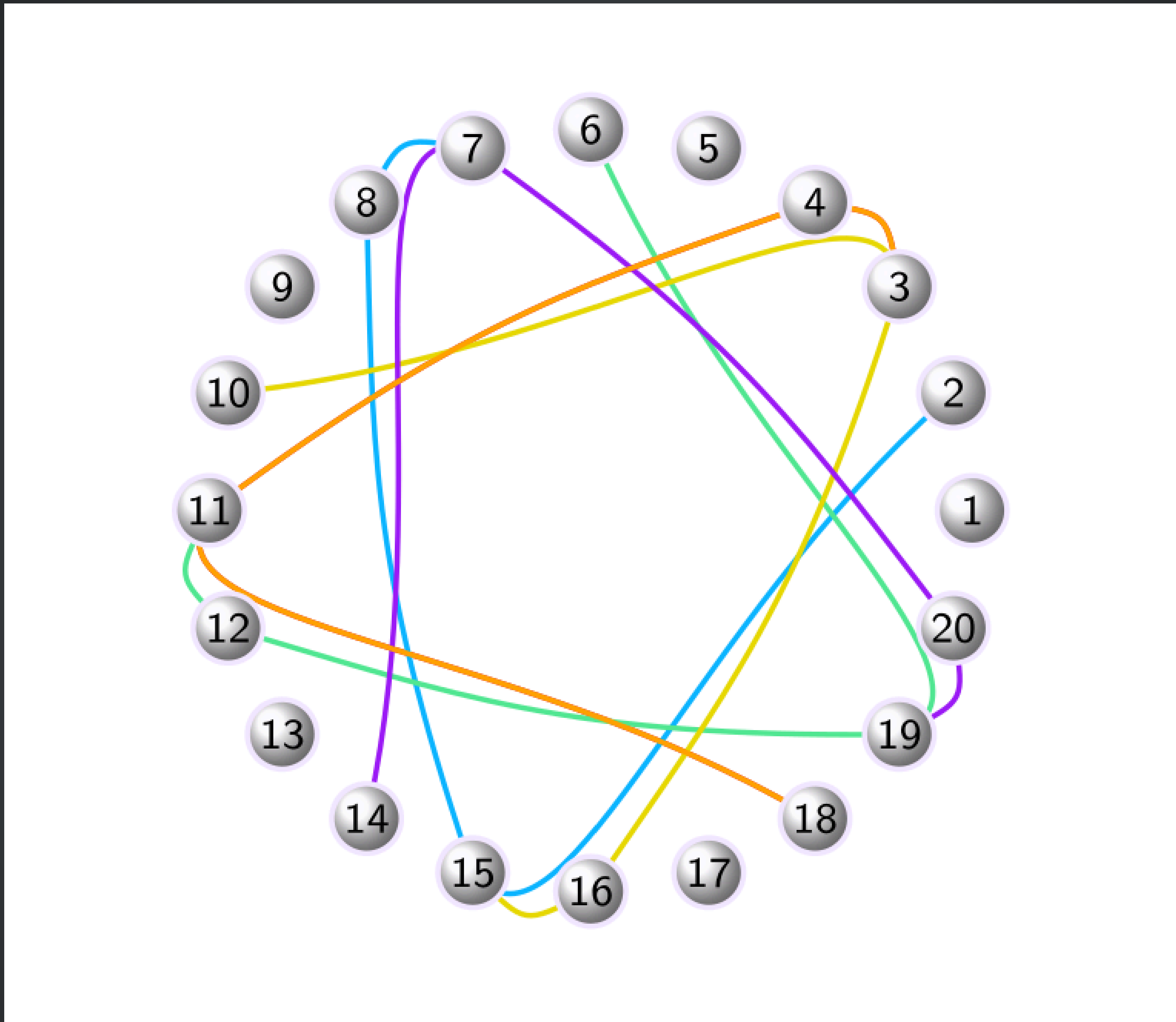
$$\mathrm{DLSub}(9,3,2,1),(1,5,9,13,17)\ldots(4,8,12,16,20) \rightarrow [1,3,9,19]$$



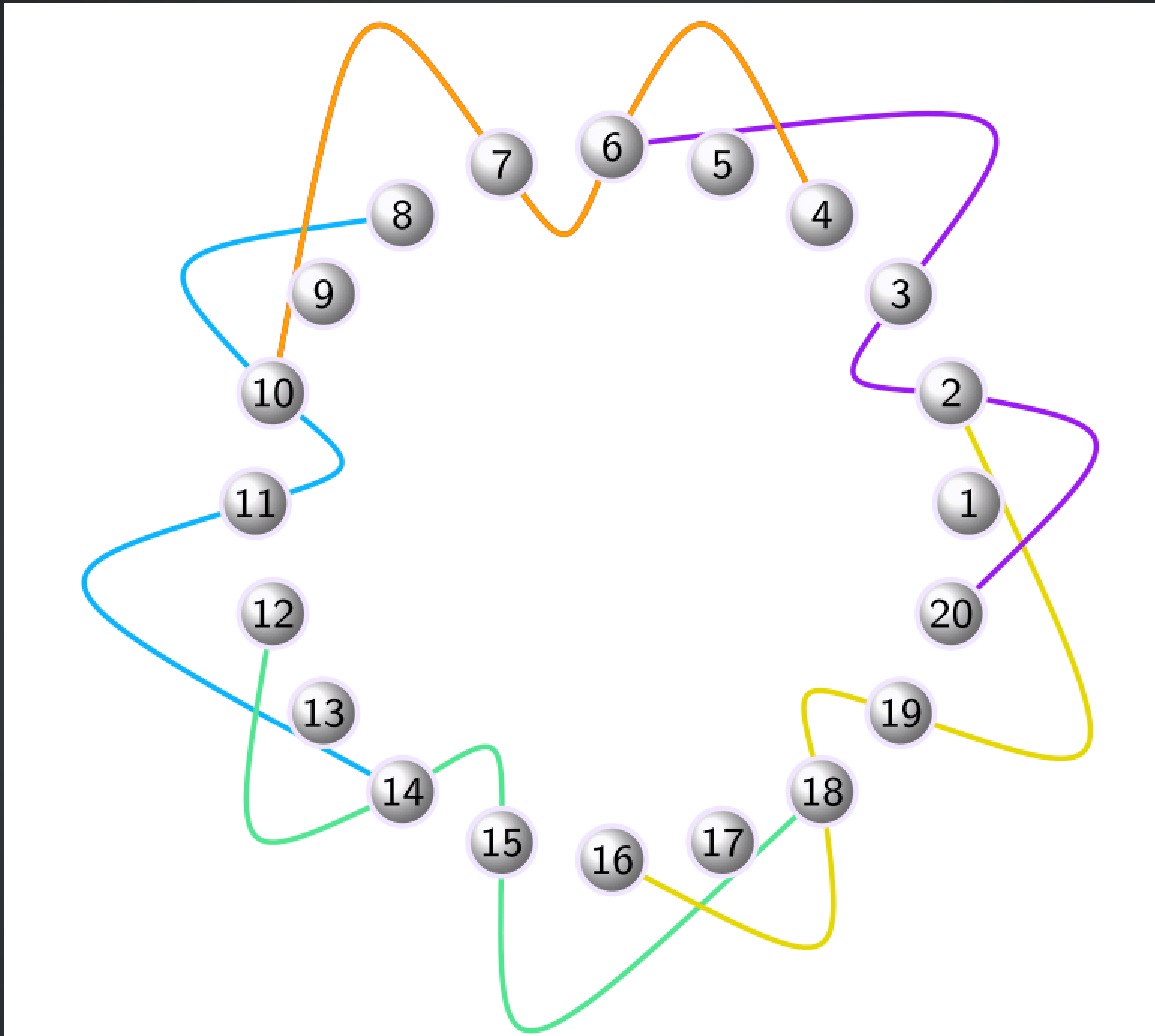
$$\mathrm{DLSub}(9,3,2,1),(1,5,9,13,17)\dots(4,8,12,16,20)\to[1,7,12,16]$$



$$\mathrm{DLSub}(9,3,2,1),(1,5,9,13,17)\ldots(4,8,12,16,20)\rightarrow[1,8,18,20]$$



$$\mathrm{DLSub}(9,3,2,1),(1,5,9,13,17)\ldots(4,8,12,16,20)\rightarrow[3,4,11,18]$$



$$\mathrm{DLSub}(9,3,2,1),(1,5,9,13,17)\ldots(4,8,12,16,20)\to[4,6,7,10]$$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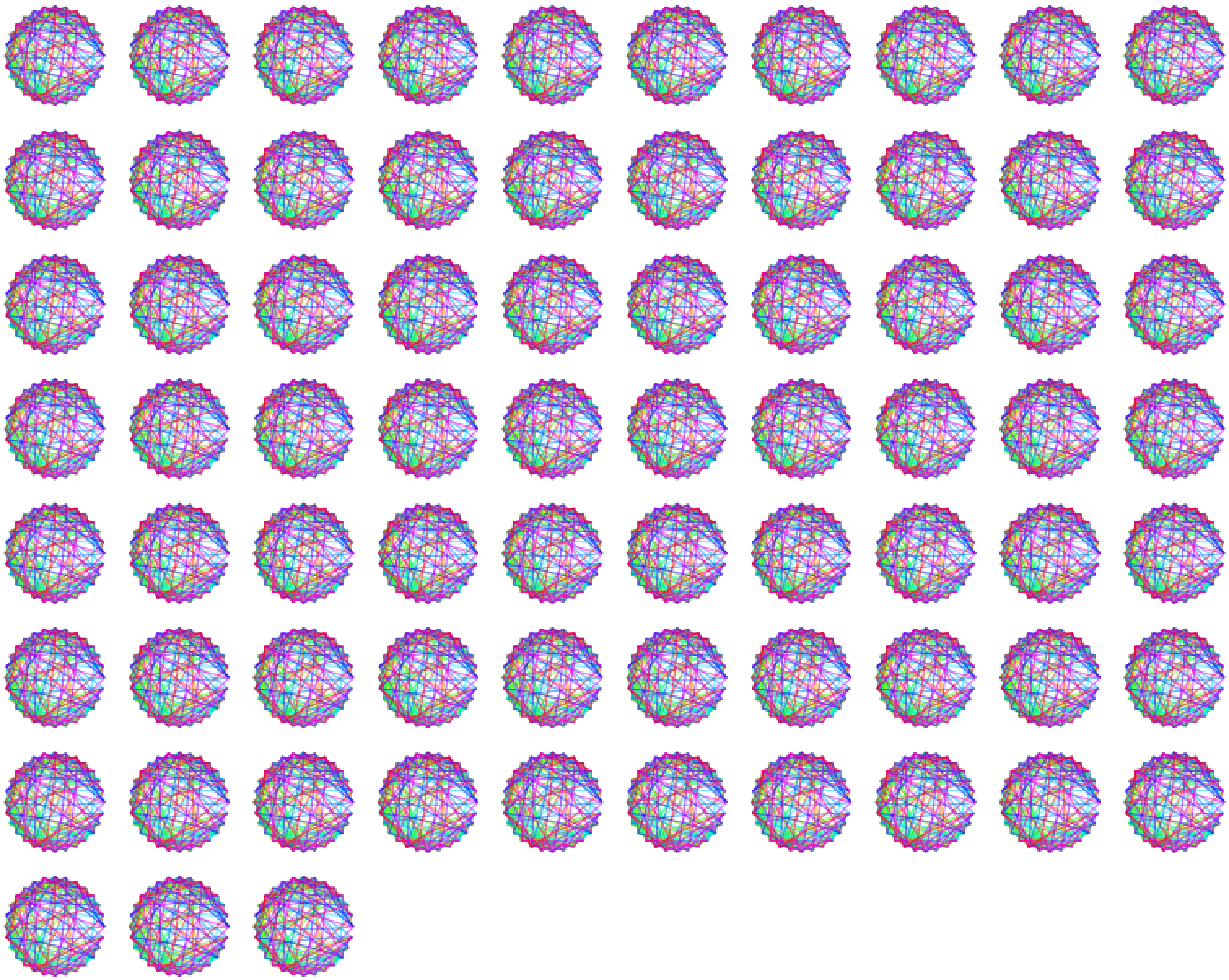
Theorem

Let $\mathcal{D}$ be a disconnected proper partial linear space admitting a group G of automorphisms satisfying Hypothesis 1. Then one of the following holds:

- $\mathcal{D}$ is a disjoint union of equal-sized lines, namely the nontrivial blocks of imprimitivity of G;
- $\mathcal{D}$ is a union of 73 copies of the $\mathrm{Ree}(3)$-unital and $G = P\Gamma L_3(8)$ acting on the set of $73 \cdot 28 = 2044$ points; each copy consists of the 28 points of a block $\sigma \in \Sigma$. In particular, $G_\sigma^\sigma \cong P\Gamma L_2(8) \cong \mathrm{Ree}(3)$ and $\mathrm{Aut}(\mathcal{D}) \cong \mathrm{Ree}(3) \wr \mathrm{Sym}(73)$.



FURTHER CLASSIFICATIONS

Let $G \leq Y \wr X$, $\quad Y = G_\sigma^\sigma$, $\quad X = G^\Sigma$ -- both 2-transitive, $\quad K = G_{(\Sigma)}$

Y is almost simple. Then G has rank 3 iff one o.f.h.:

- $G \cap soc(Y)^n = soc(Y)^n$
- G is quasiprimitive of rank 3
- handful of specific groups

Y is affine. If G has rank 3 then o.f.h.:

- G is semiprimitive of rank 3, G^Σ - almost simple
- $N \trianglelefteq G \leq N \rtimes \mathrm{Aut}(N)$, N is regular 3-orbit subgroup
- $K_{(\sigma)}$ is transitive on $\sigma' \in \Sigma \backslash \{\sigma\}$
- $K_{(\sigma)} \neq 1$ is intransitive on σ'

3-ORBIT GROUPS

A group G is a k-orbit group of $\mathrm{Aut}(G)$ has k orbits on G.

Example

1-orbit groups: $\{1\}$

2-orbit groups: elementary-Abelian p-group

k-orbit groups: $C_{p^{k-1}}, Q_{2^{k-1}}$

Lemma

If $\mathrm{Aut}(G)$ has k orbits on G, then the subgroup $\mathrm{Hol}(G) = G \rtimes \mathrm{Aut}(G)$ of the symmetric group $\mathrm{Sym}(G)$ has rank k and the stabilizer $\mathrm{Hol}(G)_1$ of 1 is $\mathrm{Aut}(G)$.

Theorem 1.1. *Let G be a finite 3-orbit group with $N = \langle G', \Phi(G)\rangle$ and $|N| = p^n$. Then $1 < N < G$ and G is isomorphic to a group in lines $1-7$ of Table 1. Moreover, the values of $V \cong G/N, A = \mathrm{Aut}(G)^V, W \cong N, B = \mathrm{Aut}(G)\!\downarrow\! W$ are valid, where $\mathrm{Aut}(G)^V$ and $\mathrm{Aut}(G)\!\downarrow\! W$ denote the groups induced on G/N and N by $\mathrm{Aut}(G)$.*

Stephen P. Glasby: arXiv:2411.11273

Table 1: 3-orbit groups G and $V \cong G/N, A = \mathrm{Aut}(G)^{G/N}, B = \mathrm{Aut}(G)\!\downarrow\! N$

G	V	A	N	B	Comments	Ref.
1. $(\mathrm{C}_{p^2})^n$	$\mathbb{F}_p^n$	$\mathrm{GL}_n(p)$	$\mathbb{F}_p^n$	$\mathrm{GL}_n(p)$	$p \geqslant 2$, G abelian	p. 4
2. $\mathbb{F}_q^d \rtimes \mathrm{C}_r$	$\mathbb{F}_r$	$\mathrm{GL}_1(r)$	$\mathbb{F}_q^d$	$\Gamma\mathrm{L}_d(q)$	$q = p^{r-1}; p \neq r, d = \frac{n}{r-1}$	6.12
3. $A(n,\theta)$	$\mathbb{F}_2^n$	$\Gamma\mathrm{L}_1(2^n)$	$\mathbb{F}_2^n$	$\Gamma\mathrm{L}_1(2^n)$	Def. 3.2(a), $n \neq 2^\ell$	6.14
4. $B(n)$	$\mathbb{F}_2^{2n}$	$\Gamma\mathrm{L}_1(2^{2n})$	$\mathbb{F}_2^n$	$\Gamma\mathrm{L}_1(2^n)$	Def. 3.2(b), $n \geqslant 1$	6.14
5. P	$\mathbb{F}_2^6$	$\mathrm{C}_7 \rtimes \mathrm{C}_9$	$\mathbb{F}_2^3$	$\Gamma\mathrm{L}_1(2^3)$	Def. 3.2(c), $n = 3$	6.14
6. $\mathbb{F}_q^3{:}\mathbb{F}_q^3$	$\mathbb{F}_q^3$	$\Gamma\mathrm{L}_3(q)$	$\mathbb{F}_q^3$	$\Gamma\mathrm{L}_3^+(q)$	$q = p^{\frac{n}{3}}$ odd, $3 \mid n$	6.9
7. $\mathbb{F}_{p^n}{:}\mathbb{F}_q^{\frac{m}{b}}$	$\mathbb{F}_q^{\frac{m}{b}}$	$\mathrm{Sp}_{\frac{m}{b}}(q) \leqslant$	$\mathbb{F}_{p^n}$	$\Gamma\mathrm{L}_1(p^n) \leqslant$	$q = p^b$ odd, $n \mid b \mid m$	6.2

Theorem B. *A finite group is a 3-orbit group if and only if it is one of the groups listed in Table 1, where p, q are distinct primes, and m, n are positive integers.*

Cai Heng Li, Yan Zhou Zhu: arXiv:2403.01725

TABLE 1. Finite 3-orbit groups

	N	$\mathrm{Aut}(N)$	Conditions	Ref
(1)	$\mathbb{Z}_p^{n(q-1)}{:}\mathbb{Z}_q$	$\mathrm{A}\Gamma\mathrm{L}(n, p^{q-1})$	Example 2.1	2.2
(2)	$\mathbb{Z}_{p^2}^n$	$\mathrm{GL}(n, \mathbb{Z}/p^2\mathbb{Z}) \cong p^{n^2}.\mathrm{GL}(n,p)$		2.3
(3)	$A_2(n,\theta) \cong 2^{n+n}$	$2^{n^2}{:}\Gamma\mathrm{L}(1,2^n)$	$\lvert\theta\rvert \neq 1$ is odd	2.6
(4)	$\mathrm{SU}(3,2^n)_2 \cong 2^{n+2n}$	$2^{2n^2}{:}\Gamma\mathrm{L}(1,2^{2n})$		2.6
(5)	$P(\epsilon) \cong 2^{3+6}$	$2^{18}{:}(\mathbb{Z}_7{:}\mathbb{Z}_9)$		2.6
(6)	$A_p(n,\theta) \cong p^{n+n}$	$p^{n^2}{:}\Gamma\mathrm{L}(3, p^{n/3})$	p is odd and $\lvert\theta\rvert = 3$	4.3
(7)	$q_+^{1+2m} \cong p^{n+2mn}$	$p^{2mn^2}{:}(\mathrm{Sp}(2m,q){:}\Gamma\mathrm{L}(1,q))$	$q = p^n$ and p is odd	4.4
(8)	$q_+^{1+2m}/U \cong p^{n_0+2mn}$	$p^{2mn_0n}{:}(\mathrm{Sp}(2m,q){:}\Gamma\mathrm{L}(1,q)_U)$	Example 5.3	5.4