Imprimitive Permutation Groups of Rank 3

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- Rank of $G \leq \mathrm{Sym}(\Omega)$ is the number of orbits
  - of G on $\Omega \times \Omega$
  - of G_α on Ω if G is transitive

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  (Huppert, Hering, Maillet, Howlet, Curtis, Kantor, Seitz, Cameron ...)
    - Affine $\qquad G \leq AGL(V)$
    - Almost simple $\qquad G_0 \leq G \leq \mathrm{Aut}(G_0)$

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- Rank 3 primitive groups, 1980's
  (Cameron, Bannai, Kantor, Liebler, Liebeck, Saxl...)
    - (i) $T \times T \trianglelefteq G \leq T_0 \wr C_2$; T-simple, $T_0 \leq Aut(T)$ - 2-tran.; $n_0^2 = n$
    - (ii) G is affine
    - (iii) G is almost simple

Imprimitive Rank 3 groups

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Known classifications:

- Quasiprimitive: $\forall$ min. normal subgr. is transitive (Plinth)

[1] [Devillers, Giudici, Li, Pearce, Praeger, 2011]

- Innately transitive: $\exists$ transitive min. normal subgroup

[2] [B., Devillers, Praeger, 2023]

- Semiprimitive*: normal subgr is either trans. or semireg.

[3] [Huang, Li, Zhu, 202?]

Classifications : QP + IT + SD

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Theorem [3]

If $G \leq \mathrm{Sym}(\Omega)$ is (imprimitive) semiprimitive of rank 3, then

a) G^{Σ} is almost simple (and known)

b) $N \trianglelefteq G \leq N \rtimes \mathrm{Aut}(N)$, N is regular special p-subgr.

In particular, $\mathrm{Aut}(N)$ has at mos 3 orbits on N.

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In particular, $Aut(N)$ has at mos 3 orbits on N.

Plan:

1. Results from the 3 papers on G^{Σ}-almost simple + Example

2. Results on N s.t. $Aut(N)$ has 3 orbits.

3. What if G^{Σ} - almost simple and no other conditions ?

M	$\lvert\Sigma\rvert$	r	G	Conditions on G	type
$\mathrm{PSL}_n(q)$	$\frac{q^n-1}{q-1}$	prime such that $r \mid (q-1)$, $o_r(p) = r-1$ and $(n,r) \neq (2,2)$	$\langle \omega^r I\rangle\, \mathrm{SL}_n(q)/\langle \omega^r I\rangle \leqslant G \leqslant \Gamma\mathrm{L}_n(q)/\langle \omega^r I\rangle$	$\lvert G^\Sigma/(G^\Sigma \cap \mathrm{PGL}_n(q))\rvert = a/j$ with $(j, r-1) = 1$	qp/it/sp
$\mathrm{PSL}_2(q)$	$q+1$	2	$\langle \omega^2 I\rangle\, \mathrm{SL}_2(q)/\langle \omega^2 I\rangle \leqslant G \leqslant \Gamma\mathrm{L}_2(q)/\langle \omega^2 I\rangle$	$q \geqslant 5$, q is odd, $G^\Sigma \not\leqslant \mathrm{P}\Sigma\mathrm{L}_2(q)$	qp/it/sp
$\mathrm{PSU}_3(q)$	q^3+1	odd prime such that $r \mid q-1$, $o_r(p) = r-1$	$\langle \omega I\rangle\, \mathrm{SU}_3(q)/\langle \omega^r I\rangle \leqslant G \leqslant \Gamma\mathrm{U}_3(q)/\langle \omega^r I\rangle$	$\lvert G^\Sigma/(G^\Sigma \cap \mathrm{PGU}_3(q))\rvert = 2a/j$ with $(j, r-1) = 1$	it
$\mathrm{PSL}_3(2)$	7	2	$\mathrm{PSL}_3(2)$, $C_2 \times \mathrm{PSL}_3(2)$		qp, it
M_{11}	11	2	M_{11}, $C_2 \times \mathrm{M}_{11}$		qp, it
$\mathrm{PSL}_3(4)$	21	6	$\mathrm{PGL}_3(4)$, $\mathrm{P\Gamma L}_3(4)$		qp, qp
$\mathrm{PSL}_3(5)$	31	5	$\mathrm{PSL}_3(5)$		qp
$\mathrm{PSL}_5(2)$	31	8	$\mathrm{PSL}_5(2)$		qp
$\mathrm{PSL}_3(8)$	73	28	$\mathrm{P\Gamma L}_3(8)$		qp
$\mathrm{PSL}_3(3)$	13	3	$\mathrm{PSL}_3(3)$		qp
$\mathrm{Alt}(6)$	6	3	$3.\mathrm{Sym}(6)$	$G_\alpha = \mathrm{Alt}(5)$	sp
M_{12}	12	2	$2.\mathrm{M}_{12}$	$G_\alpha = \mathrm{M}_{11}$	sp

TABLE 1.1. G satisfying Hypothesis 1.1, $q = p^a$ with p prime

Example: $\mathrm{Soc}(G^{\Sigma}) \cong PSL_n(q)$

5

$n \geq 2$, $q = p^a \geq 3$, $\langle \omega \rangle = \mathbb{F}_q^*$, $V = \mathbb{F}_q^n$, $\binom{V}{1}$ - set of 1-subsp.

Let r - integer div $q-1$. Define:

$\Omega = \{ \langle \omega^r \rangle u \mid u \in V^* \}$ $\qquad$ $g : \langle \omega^r \rangle u \mapsto \langle \omega^r \rangle (u) g$, $g \in \Gamma L_n(q)$

kernel $Y = \langle \omega^r I \rangle$, $\overline{G} = \Gamma L_n(q) / Y < \mathrm{Sym}(\Omega)$

$\Sigma = \{ \sigma(U) \mid U \in \binom{V}{1} \}$ where $\sigma(U) = \{ \langle \omega^r \rangle \omega^i u \mid i = 0, \ldots, r-1 \}$, $\langle u \rangle = U$

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Theorem If $SL_n(q) Y / Y \leq G \leq \overline{G}$, then:

a) G is semiprimitive on Ω

b) G is innately transitive $\Leftrightarrow r \mid (q-1)/(n, q-1)$ $[SL_n(q)Y/Y \cong PSL_n(q)]$

c) G is qusiprimitive $\Leftrightarrow r \mid (q-1)/(n, q-1)$ and $G \cap (\langle \omega I \rangle / Y) = 1$

d) G has rank 3 on $\Omega \Leftrightarrow r$ is prime and ... (see Table)

Theorem 1.1. *Let G be a finite 3-orbit group with $N = \langle G', \Phi(G)\rangle$ and $|N| = p^n$. Then $1 < N < G$ and G is isomorphic to a group in lines $1-7$ of Table 1. Moreover, the values of $V \cong G/N, A = \mathrm{Aut}(G)^V, W \cong N, B = \mathrm{Aut}(G){\downarrow}W$ are valid, where $\mathrm{Aut}(G)^V$ and $\mathrm{Aut}(G){\downarrow}W$ denote the groups induced on G/N and N by $\mathrm{Aut}(G)$.*

Table 1: 3-orbit groups G and $V \cong G/N, A = \mathrm{Aut}(G)^{G/N}, B = \mathrm{Aut}(G){\downarrow}N$

G	V	A	N	B	Comments	Ref.
1. $(\mathrm{C}_{p^2})^n$	$\mathbb{F}_p^n$	$\mathrm{GL}_n(p)$	$\mathbb{F}_p^n$	$\mathrm{GL}_n(p)$	$p \geqslant 2$, G abelian	p. 4
2. $\mathbb{F}_q^d \rtimes \mathrm{C}_r$	$\mathbb{F}_r$	$\mathrm{GL}_1(r)$	$\mathbb{F}_q^d$	$\Gamma\mathrm{L}_d(q)$	$q = p^{r-1}; p \neq r, d = \frac{n}{r-1}$	6.12
3. $A(n, \theta)$	$\mathbb{F}_2^n$	$\Gamma\mathrm{L}_1(2^n)$	$\mathbb{F}_2^n$	$\Gamma\mathrm{L}_1(2^n)$	Def. 3.2(a), $n \neq 2^\ell$	6.14
4. $B(n)$	$\mathbb{F}_2^{2n}$	$\Gamma\mathrm{L}_1(2^{2n})$	$\mathbb{F}_2^n$	$\Gamma\mathrm{L}_1(2^n)$	Def. 3.2(b), $n \geqslant 1$	6.14
5. P	$\mathbb{F}_2^6$	$\mathrm{C}_7 \rtimes \mathrm{C}_9$	$\mathbb{F}_2^3$	$\Gamma\mathrm{L}_1(2^3)$	Def. 3.2(c), $n = 3$	6.14
6. $\mathbb{F}_q^3{:}\mathbb{F}_q^3$	$\mathbb{F}_q^3$	$\Gamma\mathrm{L}_3(q)$	$\mathbb{F}_q^3$	$\Gamma\mathrm{L}_3^+(q)$	$q = p^{\frac{n}{3}}$ odd, $3 \mid n$	6.9
7. $\mathbb{F}_{p^n}{:}\mathbb{F}_q^{\frac{m}{b}}$	$\mathbb{F}_q^{\frac{m}{b}}$	$\mathrm{Sp}_{\frac{m}{b}}(q) \leqslant$	$\mathbb{F}_{p^n}$	$\Gamma\mathrm{L}_1(p^n) \leqslant$	$q = p^b$ odd, $n \mid b \mid m$	6.2

Theorem B. *A finite group is a 3-orbit group if and only if it is one of the groups listed in Table 1, where p, q are distinct primes, and m, n are positive integers.*

TABLE 1. Finite 3-orbit groups

	N	$\mathrm{Aut}(N)$	Conditions	Ref
(1)	$\mathbb{Z}_p^{n(q-1)}{:}\mathbb{Z}_q$	$\mathrm{A\Gamma L}(n, p^{q-1})$	Example 2.1	2.2
(2)	$\mathbb{Z}_{p^2}^n$	$\mathrm{GL}(n, \mathbb{Z}/p^2\mathbb{Z}) \cong p^{n^2}.\mathrm{GL}(n,p)$		2.3
(3)	$A_2(n,\theta) \cong 2^{n+n}$	$2^{n^2}{:}\Gamma\mathrm{L}(1, 2^n)$	$\|\theta\| \neq 1$ is odd	2.6
(4)	$\mathrm{SU}(3, 2^n)_2 \cong 2^{n+2n}$	$2^{2n^2}{:}\Gamma\mathrm{L}(1, 2^{2n})$		2.6
(5)	$P(\epsilon) \cong 2^{3+6}$	$2^{18}{:}(\mathbb{Z}_7{:}\mathbb{Z}_9)$		2.6
(6)	$A_p(n,\theta) \cong p^{n+n}$	$p^{n^2}{:}\Gamma\mathrm{L}(3, p^{n/3})$	p is odd and $\|\theta\| = 3$	4.3
(7)	$q_+^{1+2m} \cong p^{n+2mn}$	$p^{2mn^2}{:}(\mathrm{Sp}(2m,q){:}\Gamma\mathrm{L}(1,q))$	$q = p^n$ and p is odd	4.4
(8)	$q_+^{1+2m}/U \cong p^{n_0+2mn}$	$p^{2mn_0n}{:}(\mathrm{Sp}(2m,q){:}\Gamma\mathrm{L}(1,q)_U)$	Example 5.3	5.4

Can we classify all G of rank 3 with G^{Σ}-almost simple? (7)

Let $G \leq Y \wr X$, $\overset{\deg r}{Y = G_{\sigma}^{\sigma}}$, $\overset{\deg n}{X = G^{\Sigma}}$ - both 2-transitive.

Let $T = \mathrm{soc}(Y)$, $L = G \cap T^{n}$, $K = G_{(\Sigma)}$

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Theorem [1]. Assume T is non-abelian simple group.
G has rank 3 $\iff$ one of the following holds:

1) $L = T^n$
2) G is qusiprimitive of rank 3
3) $n = 2$, $G = M_{10}$, $PGL_2(9)$ or $Aut(A_6)$ acting on 12 points
4) $n = 2$, $G = Aut(M_{12})$ acting on 24 points

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Theorem [3]. Assume Y is affine ($T \simeq \mathbb{F}_p^d$), G has rank 3 =>

1) G is semiprim. with G^{Σ} - almost simple
2) $N \trianglelefteq G \leq N \rtimes Aut(N)$, N is regular 3-orbit subgroup.
3) $K_{(\sigma)}$ is transitive on $\sigma' \in \Sigma \setminus \{\sigma\}$
4) $K_{(\sigma)} \neq 1$ is intransitive on σ'; G has el-ab self-centr. normal subgroup. (L)

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Theorem [3]. Assume Y is affine ($T \cong \mathbb{F}_p^d$), G has rank 3 $\Rightarrow$

1) G is semiprim. with G^Σ-almost simple V

G^Σ-affine

2) ~~$N \le G \le N \rtimes Aut(N)$, N is regular 3-orbit subgroup.~~
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$\to C_G(L) = L$

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$Y = T \rtimes Y_0$

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$$\left.\begin{array}{l} \bullet\ C_G(L) = L \\ C_G(T^n) \leq C_G(L) = L \end{array}\right\} \Rightarrow \underbrace{G/L}_{\overline{G}} \lesssim \mathrm{Aut}(T^n) = GL_{nd}(\mathbb{F}_p)$$

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- $C_G(L) = L$
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$$\Rightarrow \quad \overline{G} = G/L \lesssim Aut(T^n) = GL_{nd}(\mathbb{F}_p)$$

- Basis $\beta = \beta_1 \cup \ldots \cup \beta_n$ of $V = \mathbb{F}_p^{nd}$
- $\overline{G} \leq Y_0 \wr X \leq GL_{nd}(\mathbb{F}_p)$

$$g \sim \begin{pmatrix} & \boxed{g_1} & & \\ & & \boxed{g_2} & \\ & & & \ddots \\ \boxed{g_3} & & & \end{pmatrix} \qquad g_i \in Y_i \leq GL_d(\mathbb{F}_p)$$

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Let $T = \mathrm{soc}(Y) \simeq \mathbb{F}_p^d$, $\quad L = G \cap T^n$, $\quad K = G_{(\Sigma)}$

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$L \leq V$ is $\overline{G}$-invariant subspace. What are the options?

$\overline{G}$ is monomial

- $d=1$, $\overline{G} \le M_n(\mathbb{F}_p)$ – subgroup of monomial matr. in $GL_n(\mathbb{F}_p)$
- $\varphi : M_n(p) \to Sym(n)$
- $\varphi(\overline{G}) = X$ – 2-transitive almost simple.

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Lemma

If $\overline{G} \cap D_n(p) \nleq Z(GL_n(p))$ then $\overline{G}$ is irreducible.

- $\Rightarrow L = V = T^n$, so $L(\sigma) \leq K(\sigma)$ is transitive on σ' and G' has rank 3

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If $\overline{G} \le ZX$, then $\overline{G}$ leaves invariant the following:

1) $U = \langle (1, 1, \dots, 1) \rangle$ of dim 1
2) $U^{\perp} = \{ (a_1, \dots a_n) \mid \sum a_i = 0 \}$ of dim $n-1$.
3) Sometimes something else see B. Mortimer paper

"The modular perm. rep. of known 2-trans. groups" 1980

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If $\overline{G} \leq ZX$, then $\overline{G}$ leaves invariant the following:

1) $U = \langle (1, 1, \dots, 1) \rangle$ of dim 1 not rank 3
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"The modular perm. rep. of known 2-trans. groups" 1980

Table 1. Reducibility of the hearts of the known 2-transitive groups

Group, G	Degree $\|\Omega\|$	Transitivity	Conditions under which the heart of G over K is reducible (K a field of characteristic p)
$\mathrm{Sym}(n)$, $n \geqslant 3$	n	n	always simple
$\mathrm{Alt}(n)$, $n \geqslant 5$	n	$n-2$	always simple
$\mathrm{Alt}(4) = \mathrm{AGL}(1,4)$	4	2	$K \geqslant F_4$
$G \leqslant \mathrm{A\Gamma L}(d,q)$ containing the translations	q^d	2 or 3	p divides q
$\mathrm{PSL}(d,q) \leqslant G \leqslant \mathrm{P\Gamma L}(d,q)$, $d \geqslant 3$	$(q^d-1)/(q-1)$	2	p divides q
$G \cong \mathrm{Alt}(7) < \mathrm{PGL}(4,2)$	15	2	$p = 2$
$[\mathrm{Sp}(2m,2)]^-$, $m \geqslant 3$	$2^{m-1}(2^m-1)$	2	$p = 2$
$[\mathrm{Sp}(2m,2)]^+$, $m \geqslant 2$	$2^{m-1}(2^m+1)$	2	$p = 2$
G, a 3-transitive subgroup of $\mathrm{P\Gamma L}(2,q)$	$q+1$	3	always simple
$\mathrm{PSL}(2,q) \leqslant G \leqslant \mathrm{P\Sigma L}(2,q)$	$q+1$	2	$K \geqslant F_2$ if $q \equiv \pm 1 \pmod 8$; $K \geqslant F_4$ if $q \equiv \pm 3 \pmod 8$
$\mathrm{Sz}(q) \leqslant G \leqslant \mathrm{Aut}(\mathrm{Sz}(q))$	q^2+1	2	p divides $q+1+m$ where $m^2 = 2q$
$\mathrm{PSU}(3,q^2) \leqslant G \leqslant \mathrm{P\Gamma U}(3,q^2)$	q^3+1	2	p divides $q+1$
$\mathrm{Re}(q) \leqslant G \leqslant \mathrm{Aut}(\mathrm{Re}(q))$	q^3+1	2	p divides $(q+1)(q+m+1)$ and perhaps if p divides $(q-m+1)$ where $m^2 = 3q$
M_{24}	24	5	$p = 2$
M_{23}	23	4	$p = 2$
M_{22}	22	3	$p = 2$
M_{12}	12	5	always simple
M_{11}	11	4	always simple
M_{11}	12	3	$p = 3$
$\mathrm{PSL}(2,11)$	11	2	$p = 3$
HS	176	2	$p = 2, 3$
CO_3	276	2	perhaps if $p = 2$ or 3

$\overline{G}$ is monomial

Conjecture.

If $\overline{G} \cap D_n(p) \leq Z$, then $\overline{G}$ is conjugate to a subgroup of $Z \cdot \mathrm{Per}_n(p)$ by an element from $M_n(p)$.