A REINFORCEMENT LEARNING APPROACH TO COMPUTING ZETA FUNCTIONS OF GROUPS AND ALGEBRAS

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GOAL

Goal: effectively compute local **subobject** zeta functions of (Lie) algebras:

Let k be a number field with ring of integers $\mathfrak{o}$.

Let A be an $\mathfrak{o}$-algebra, free of rank d as an $\mathfrak{o}$-module.

$$\zeta_A^{\leq}(s) = \sum_{U \leq A} |A : U|^{-s} = \prod_{v \in \mathcal{V}_k} \zeta_{A_v}^{\leq}(s)$$

where $A_v = A \otimes_{\mathfrak{o}} \mathfrak{o}_v$.

Example $L = \mathrm{Fil}_4$

$\langle z, y_1, y_2, y_3, y_4 \mid [z, y_1] = y_2, [z, y_2] = y_3, [z, y_3] = y_4, [y_1, y_2] = y_4 \rangle$.

Additive rank 5, nilpotency class 4 $\mathbb{Z}$-Lie algebra.

RATIONALITY AND UNIFORMITY

Let L be a ring of finite additive rank.

Theorem (Grunewald,Segal,Smith - 1988)

Let p be prime. The local zeta function $\zeta_{L,p}(s)$ is a rational function in p^{-s}. i.e. there is a function $W_p(Y) = P_p(Y)/Q_p(y) \in \mathbb{Q}(Y)$ such that

$$\zeta_{L,p}(s) = W_p(p^{-s}).$$

Theorem (du Sautoy, Grunewald - 2000)

There are smooth algebraic varieties V_t, $t \in \{1, \dots, m\}$ defined over $\mathbb{Q}$, and $W_t(X, Y) \in \mathbb{Q}(X, Y)$ such that, for almost all p,

$$\zeta_{L,p}(s) = \sum_{t=1}^{m} c_t(p) W_t(p, p^{-s}),$$

where $c_t(p)$ is the number of $\mathbb{F}_p$-rational points of $\overline{V_t}$.

RATIONALITY AND UNIFORMITY

Let L be a ring of finite additive rank.

- $\zeta_L(s)$ is **uniform** if there exists a rational function $W(X, Y)$ such that

  $$\zeta_{L,p}(s) = W(p, p^{-s})$$

  for all p.
- $\zeta_L(s)$ is **almost uniform** if there exists a rational function $W(X, Y)$ such that

  $$\zeta_{L,p}(s) = W(p, p^{-s})$$

  for **almost all** p. In this case, $W(X, Y) \in \mathbb{Q}(X, Y)$ is the **generic local zeta function** of L.

p-ADIC INTEGRATION

For $f(\mathbf{x}) \in \mathbb{Z}[x_1, \ldots, x_n]$

$$\int_{\mathbb{Z}_p^n} |f(x)|_p^s \, \mathrm{d}\mu = \sum_{m=0}^{\infty} \mu\left(\{\mathbf{x} \in \mathbb{Z}_p^n \mid v_p(f(\mathbf{x})) = m\}\right) p^{-sm}$$

- $\mathbb{Z}_p$ – p-adic integers, $|\cdot|_p$ – p-adic absolute value
- μ is the Haar measure normalised s.t. $\mu(\mathbb{Z}_p^n) = 1$.
- $\mu(a + S) = \mu(S)$ for $S \subseteq \mathbb{Z}_p^n$
- $\mu(a + p^n\mathbb{Z}_p) = \mu(p^n\mathbb{Z}_p) = p^{-n}$

p-ADIC INTEGRATION

- Let $L = \langle x, y, z \mid [x, y] = z, [x, z] = [y, z] = 0\rangle$ (Heisenberg)
- Local zeta function: need to count subalgebras in the $\mathbb{Z}_p$ -algebra L_p.
- The rows of a matrix $M = (x_{ij}) \in \mathrm{Tr}_3(\mathbb{Z}_p) \cap \mathrm{GL}_3(\mathbb{Q}_p)$ – generators of a full additive sublattice Λ of $\mathbb{Z}_p^3$.
- For Λ to be a subalgebra it has to be closed under taking Lie brackets of generators: $x_{33} | x_{11} x_{22}$.

$$\begin{aligned}\zeta_{L_p}(s) &= \sum_{\Lambda \leq L_p} |L_p : \Lambda|^{-s} \\ &= (1-p^{-1})^{-3} \int_{\{\mathbf{x} \in \mathrm{Tr}_3(\mathbb{Z}_p) : x_{33} | x_{11} x_{22}\}} |x_{11}|_p^{s-1} |x_{22}|_p^{s-2} |x_{33}|_p^{s-3} d\mu^{(6)}\end{aligned}$$

Three basic ingredients of **cone integral**:

- integrand
- area of integration $\mathrm{Tr}_3(\mathbb{Z}_p) \cong \mathbb{Z}_p^6$
- polynomial divisibility conditions (**cone conditions**)

EXAMPLE

Let $L = \mathrm{Fil}_4 = \langle z, y_1, y_2, y_3, y_4 \mid [z, y_1] = y_2, [z, y_2] = y_3, [z, y_3] = y_4, [y_1, y_2] = y_4 \rangle$. Additive rank 5, nilpotency class 4.

$$\zeta_{L_p}(s) = (1 - p^{-1})^{-5} \int_{\mathbf{x} \in \mathbb{Z}_p^{15} : C} |x_0|_p^{1-s} |x_5|_p^{2-s} |x_9|_p^{3-s} |x_{12}|_p^{4-s} |x_{14}|_p^{5-s} d\mu$$

$$\begin{aligned} C = x_{12}x_{14}x_9 &| - x_0x_{10}x_{13}x_5 + x_0x_{11}x_{12}x_5 - x_0x_{12}x_7x_9 + x_0x_{13}x_6x_9 \\ &- x_1x_{12}x_6x_9 + x_{12}x_2x_5x_9, \\ x_{12}x_{14} &| x_0x_{10}x_{12} - x_0x_{13}x_9 + x_1x_{12}x_9, \\ x_{12}x_9 &| - x_0x_{10}x_5 + x_0x_6x_9, \\ x_9 &| x_0x_5, x_{12} | x_0x_9, x_{14} | x_0x_{12}, x_{14} | x_5x_9 \end{aligned}$$

HOW DO I CALCULATE THAT???

Option 1: by hand

Many people: using various tricks and instruments from algebraic geometry (blow-ups, smart changes of variables etc.) **reduce conditions to monomial**.

For example, Woodward (2005):

(★) Trivial: the integral reduces to monomials straight away.

(★★) Easy: no more than three blow-ups are needed to reduce this one to monomials.

(★★★) Medium: a few more blow-ups are needed, but it should be fairly easy to see the way through the calculation.

(★★★★) Hard: considerably more blow-ups, and probably one or two entirely non-obvious tricks, will be needed if we are to complete the calculation.

Unless stated otherwise, all local zeta functions given are for all primes p.

Option 2: use computer

Package `Zeta` for SageMath by Rossmann.

`zeta` AND LIMITATIONS

`Zeta` uses tools from convex geometry and algebraic geometry (toric varieties, Gröbner bases, cone integrals).

It splits p-adic integral in a sum of cone integrals in a smart non-trivial way:

- simplify (add mon. conditions to **cone**, find better but equivalent conditions...)
- balance (cut **cone** so that initial forms a constant on each piece)
- reduce (repair certain specific types of singularity)

Most common limitations of `Zeta`:

1. Toric singularities
2. Too many rational functions to add

Idea

Both problems can be eliminated by "pre-treating" the p-adic integral using "human guidance" and ad hoc approaches.

EXAMPLE

Let $L = \mathrm{Fil}_4 = \langle z, y_1, y_2, y_3, y_4 \mid [z, y_1] = y_2, [z, y_2] = y_3, [z, y_3] = y_4, [y_1, y_2] = y_4 \rangle$. Additive rank 5, nilpotency class 4.

$$\zeta_{L_p}(s) = (1 - p^{-1})^{-5} \int_{\mathbf{x} \in \mathbb{Z}_p^{15} : C} |x_0|_p^{1-s} |x_5|_p^{2-s} |x_9|_p^{3-s} |x_{12}|_p^{4-s} |x_{14}|_p^{5-s} d\mu$$

$$\begin{aligned} C = x_{12}x_{14}x_9 &\mid -x_0x_{10}x_{13}x_5 + x_0x_{11}x_{12}x_5 - x_0x_{12}x_7x_9 + x_0x_{13}x_6x_9 \\ &\quad - x_1x_{12}x_6x_9 + x_{12}x_2x_5x_9, \\ x_{12}x_{14} &\mid x_0x_{10}x_{12} - x_0x_{13}x_9 + x_1x_{12}x_9, \\ x_{12}x_9 &\mid -x_0x_{10}x_5 + x_0x_6x_9, \\ x_9 &\mid x_0x_5, x_{12} \mid x_0x_9, x_{14} \mid x_0x_{12}, x_{14} \mid x_5x_9 \end{aligned}$$

```
In [2]: L = Zeta.lookup("Fil4")

In [3]: T = L.toric_datum("subalgebras")

In [4]: print(T)
Toric datum
Base ring: Multivariate Polynomial Ring in x0, x1, x2, x3, x4, x5, x6, x7, x8, x9, x10,
x11, x12, x13, x14 over Rational Field

Polyhedron: A 15-dimensional polyhedron in QQ^15 defined as the convex hull of 1 vertex and
23 rays
H-representation
begin
 19 16 rational
 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0
 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0
 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0
 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0
 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0
 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0
 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 -1
 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0
 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0
 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0
 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0
 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 -1
 0 1 0 0 0 0 0 0 0 0 1 0 0 -1 0 0
 0 1 0 0 0 0 1 0 0 0 -1 0 0 0 0 0
end

Integrand: t^(y * (1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 1)) * q^(y * (0, -1, -1, -1,
-1, 1, -1, -1, -1, 2, -1, -1, 3, -1, 4))
( 0 )        x12*x14  ||  x1*x9*x12 + x0*x10*x12 - x0*x9*x13
( 1 )         x9*x12  ||  x0*x6*x9 - x0*x5*x10
( 2 )     x9*x12*x14  ||  x2*x5*x9*x12 - x1*x6*x9*x12 - x0*x7*x9*x12 + x0*x5*x11*x12 +
x0*x6*x9*x13 - x0*x5*x10*x13
```

HUMANS COULD NOT DO IT

Woodward (2005):

Proposition 3.19 *Up to tensoring with* $\mathbb{R}$, *all Lie rings of rank* ≤ 5 *except perhaps* Fil_4 *have distinct zeta functions counting all subrings.*

Indeed, Fil_4 is the only Lie ring of rank 5 for which we don't know the zeta function counting all subrings. Regrettably, I have been unable to calculate this zeta function.

MACHINES COULD NOT DO IT

Rossmann (2015):

9.1. Five-dimensional nilpotent Lie algebras: Fil_4

As in [36, Thm 3.6], let Fil_4 be the nilpotent Lie ring with $\mathbf{Z}$-basis $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_5)$ and Lie bracket $[\boldsymbol{e}_1, \boldsymbol{e}_2] = \boldsymbol{e}_3$, $[\boldsymbol{e}_1, \boldsymbol{e}_3] = \boldsymbol{e}_4$, $[\boldsymbol{e}_1, \boldsymbol{e}_4] = \boldsymbol{e}_5$, $[\boldsymbol{e}_2, \boldsymbol{e}_3] = \boldsymbol{e}_5$, and $[\boldsymbol{e}_i, \boldsymbol{e}_j] = 0$ for $i \le j$ not listed above. As we explained in [27, §7.3], with the sole exception of $\mathrm{Fil}_4 \otimes_{\mathbf{Z}} \mathbf{C}$, each of the 16 isomorphism classes of non-trivial nilpotent Lie $\mathbf{C}$-algebras of dimension at most 5 admits a $\mathbf{Z}$-form whose local subring zeta functions have been computed. We can use TopologicalZetaFunction (Algorithm 4.1) and Zeta to confirm that for the 15 known types, the topological zeta function coincides with the one deduced from p-adic formulae using the informal method from the introduction. The local subring zeta functions of Fil_4 have so far resisted attempts at computing them [36, p. 57]. In [27,

BUT TOGETHER...

Humans

```
bl21 = trick_change_var(bl21, {x6:x6*x12}) # since x12 divides x6 // THIS WILL
 ↪CHANGE THE INTEGRAND
bl21 = trick_divide_by(bl21, x12)
print('\n', *bl21, sep = '\n')
del bl21[1]


bl21 = trick_change_var(bl21, {x7:x7+ x5*x11})
print('\n', *bl21, sep = '\n')
```

```
(x12*x14, x0*x10*x12 - x9*x13)
(x12, -x5*x10 + x6)
(x12*x14, x2*x5*x9*x12 - x1*x5*x10*x12 + x0*x5*x11*x12 - x0*x7*x12 - x5*x10*x13
+ x6*x13)
(x12, x9)
(x14, x0^2*x12)
(x14, x5*x9^2)


(x12*x14, x0*x10*x12 - x9*x13)
(x12, x6)
(x12*x14, x2*x5*x9*x12 - x1*x5*x10*x12 + x0*x5*x11*x12 - x0*x7*x12 + x6*x13)
(x12, x9)
(x14, x0^2*x12)
(x14, x5*x9^2)


(x12*x14, x0*x10*x12 - x9*x13)
(1, x6)
(x14, x2*x5*x9 - x1*x5*x10 + x0*x5*x11 - x0*x7 + x6*x13)
(x12, x9)
(x14, x0^2*x12)
(x14, x5*x9^2)


(x12*x14, x0*x10*x12 - x9*x13)
(x14, x2*x5*x9 - x1*x5*x10 - x0*x7 + x6*x13)
(x12, x9)
(x14, x0^2*x12)
(x14, x5*x9^2)
```

Machines

```
2025-04-30 13:32:36 Zeta.abstract INFO - POP [#unprocessed =  4, #regular = 41]
2025-04-30 13:32:36 Zeta.abstract INFO - POP [#unprocessed =  3, #regular = 42]
2025-04-30 13:32:36 Zeta.abstract INFO - POP [#unprocessed =  2, #regular = 43]
2025-04-30 13:32:36 Zeta.abstract INFO - POP [#unprocessed =  1, #regular = 44]
2025-04-30 13:32:37 Zeta.abstract INFO - Main loop finished successfully.
2025-04-30 13:32:37 Zeta.abstract INFO - Total number of regular data: 45
2025-04-30 13:32:37 Zeta.abstract INFO - Launching 4 evaluators.
2025-04-30 14:00:30 Zeta.abstract INFO - Evaluator #2 finished.
2025-04-30 14:12:43 Zeta.abstract INFO - Evaluator #3 finished.
2025-04-30 14:20:56 Zeta.abstract INFO - Evaluator #1 finished.
2025-04-30 14:31:56 Zeta.abstract INFO - Evaluator #0 finished.
2025-04-30 14:31:56 Zeta.abstract INFO - All evaluators finished.
2025-04-30 14:31:56 Zeta.addmany INFO - Total number of rational functions:
97511
2025-04-30 14:33:48 Zeta.addmany INFO - Total number of rational functions:
18320
2025-04-30 14:40:17 Zeta.abstract INFO - Total number of rational functions:
16973
2025-04-30 14:40:24 Zeta.abstract INFO - Degree of denominator in (t,q):
(930,1606)
2025-04-30 14:40:24 Zeta.addmany INFO - addmany dispatcher: numerator
2025-04-30 14:40:31 Zeta.addmany INFO - Launching 4 summators.
2025-04-30 16:48:18 Zeta.addmany INFO - Summator #1 finished.
2025-04-30 16:48:20 Zeta.addmany INFO - Summator #3 finished.
2025-04-30 16:49:35 Zeta.addmany INFO - Summator #0 finished.
2025-04-30 16:54:19 Zeta.addmany INFO - Summator #2 finished.
2025-04-30 16:54:21 Zeta.addmany INFO - All summators finished.
2025-04-30 18:14:48 Zeta.abstract INFO - Factoring final result.
2025-04-30 18:14:49 Zeta.abstract INFO - All done.
```

$$\zeta_{\mathrm{Fil}_4,p}(s) = W(p, p^{-s})$$

$$W(q,t) =$$

```
-(q^138*t^81 + q^136*t^79 + q^135*t^79 - q^135*t^78 + q^134*t^78 + 2*q^133*t^78 - 2*q^133*t^77 + 3*q^132*t^77 - 2*q^132*t^76 + 2*q^131*t^77 - 4*q^131*t^76 + q^131*t^75 +
2*q^130*t^76 - 2*q^130*t^75 - 3*q^129*t^75 + q^129*t^74 + 5*q^128*t^75 - 6*q^128*t^74 + q^128*t^73 - 5*q^127*t^74 + q^127*t^73 + 4*q^126*t^74 - 7*q^126*t^73 + 2*q^126*t^72 -
7*q^125*t^73 + 2*q^125*t^72 + q^124*t^73 - 8*q^124*t^72 + 6*q^124*t^71 - 5*q^123*t^72 + q^123*t^71 - 8*q^122*t^71 + 8*q^122*t^70 - 7*q^121*t^71 + 4*q^121*t^70 -
10*q^120*t^70 + 14*q^120*t^69 - 9*q^119*t^70 - 2*q^120*t^68 + 8*q^119*t^69 - q^118*t^70 + q^119*t^68 - 4*q^118*t^69 + 15*q^118*t^68 - 7*q^117*t^69 - 4*q^118*t^67 +
9*q^117*t^68 - q^117*t^67 - 6*q^116*t^68 + 20*q^116*t^67 - 6*q^115*t^68 - 7*q^116*t^66 + 9*q^115*t^67 - 6*q^114*t^67 + 25*q^114*t^66 - 8*q^113*t^67 - 11*q^114*t^65 +
19*q^113*t^66 - 7*q^113*t^65 - 3*q^112*t^66 + 23*q^112*t^65 - 2*q^111*t^66 - 15*q^112*t^64 + 12*q^111*t^65 - 8*q^111*t^64 - 3*q^110*t^65 - q^111*t^63 + 24*q^110*t^64 -
3*q^109*t^65 - 20*q^110*t^63 + 16*q^109*t^64 + q^110*t^62 - 14*q^109*t^63 - 4*q^108*t^64 + q^109*t^62 + 24*q^108*t^63 - q^107*t^64 - 22*q^108*t^62 + 12*q^107*t^63 +
2*q^108*t^61 - 17*q^107*t^62 - q^106*t^63 + 2*q^107*t^61 + 18*q^106*t^62 - 25*q^106*t^61 + 13*q^105*t^62 + 5*q^106*t^60 - 19*q^105*t^61 - 3*q^104*t^62 + 2*q^105*t^60 +
19*q^104*t^61 + q^105*t^59 - 23*q^104*t^60 + 11*q^103*t^61 + 5*q^104*t^59 - 17*q^103*t^60 - q^102*t^61 + 5*q^103*t^59 + 14*q^102*t^60 + q^103*t^58 - 25*q^102*t^59 +
14*q^101*t^60 + 8*q^102*t^58 - 22*q^101*t^59 + 2*q^100*t^60 + 7*q^101*t^58 + 12*q^100*t^59 + 3*q^101*t^57 - 20*q^100*t^58 + 8*q^99*t^59 + 4*q^100*t^57 - 12*q^99*t^58 +
5*q^99*t^57 + 16*q^98*t^58 + 4*q^99*t^56 - 24*q^98*t^57 + 8*q^97*t^58 + 7*q^98*t^56 - 19*q^97*t^57 + q^96*t^58 - q^98*t^55 + 5*q^97*t^56 + 12*q^96*t^57 + 7*q^97*t^55 -
25*q^96*t^56 + 9*q^95*t^57 + 6*q^96*t^55 - 14*q^95*t^56 + 3*q^95*t^55 + 9*q^94*t^56 + 3*q^95*t^54 - 21*q^94*t^55 + 5*q^93*t^56 + 4*q^94*t^54 - 6*q^93*t^55 - q^94*t^53 -
9*q^93*t^54 + 6*q^92*t^55 + 8*q^93*t^53 - 36*q^92*t^54 + 5*q^91*t^55 - q^93*t^52 + 12*q^92*t^53 - 12*q^91*t^54 + 2*q^92*t^52 - 7*q^91*t^53 + 3*q^90*t^54 + 10*q^91*t^52 -
32*q^90*t^53 + q^89*t^54 + 13*q^90*t^52 - 6*q^89*t^53 + 2*q^90*t^51 - 6*q^89*t^52 - q^88*t^53 + 13*q^89*t^51 - 27*q^88*t^52 + q^87*t^53 - 3*q^89*t^50 + 14*q^88*t^51 -
9*q^87*t^52 + 3*q^88*t^50 - 15*q^87*t^51 + q^86*t^52 + 25*q^87*t^50 - 31*q^86*t^51 - 2*q^87*t^49 + 23*q^86*t^50 - 8*q^85*t^51 + 3*q^86*t^49 - 7*q^85*t^50 - 2*q^84*t^51 +
30*q^85*t^49 - 20*q^84*t^50 - 4*q^85*t^48 + 31*q^84*t^49 - 9*q^83*t^50 - 2*q^84*t^48 + 2*q^83*t^49 - q^82*t^50 - 3*q^84*t^47 + 28*q^83*t^48 - 17*q^82*t^49 - 6*q^83*t^47 +
23*q^82*t^48 - 4*q^81*t^49 + 4*q^82*t^47 - 5*q^81*t^48 - q^80*t^49 - 4*q^82*t^46 + 37*q^81*t^47 - 15*q^80*t^48 - 10*q^81*t^46 + 32*q^80*t^47 - 6*q^79*t^48 + q^81*t^45 -
2*q^80*t^46 + 10*q^79*t^47 - q^78*t^48 - 5*q^80*t^45 + 32*q^79*t^46 - 10*q^78*t^47 - 19*q^79*t^45 + 27*q^78*t^46 - q^77*t^47 - q^79*t^44 - 7*q^78*t^45 + 4*q^77*t^46 -
5*q^78*t^44 + 24*q^77*t^45 - 8*q^76*t^46 - 19*q^77*t^44 + 16*q^76*t^45 - q^75*t^46 - 2*q^77*t^43 - 2*q^76*t^44 + 4*q^75*t^45 - 5*q^76*t^43 + 27*q^75*t^44 - 8*q^74*t^45 -
26*q^75*t^43 + 18*q^74*t^44 - 2*q^73*t^45 - 9*q^74*t^43 + 3*q^73*t^44 - 5*q^74*t^42 + 23*q^73*t^43 - 3*q^72*t^44 - 26*q^73*t^42 + 7*q^72*t^43 - q^73*t^41 - 7*q^72*t^42 -
2*q^71*t^43 - 4*q^72*t^41 + 20*q^71*t^42 - q^70*t^43 - 20*q^71*t^41 + 9*q^70*t^42 + 2*q^71*t^40 - 2*q^70*t^41 - 4*q^69*t^42 - 8*q^70*t^40 + 24*q^69*t^41 - q^68*t^42 + q^70*t^39
- 24*q^69*t^40 + 8*q^68*t^41 + 4*q^69*t^39 + 2*q^68*t^40 - 2*q^67*t^41 - 9*q^68*t^39 + 20*q^67*t^40 + q^68*t^38 - 20*q^67*t^39 + 4*q^66*t^40 + 2*q^67*t^38 + 7*q^66*t^39 +
q^65*t^40 - 7*q^66*t^38 + 26*q^65*t^39 + 3*q^66*t^37 - 23*q^65*t^38 + 5*q^64*t^39 - 3*q^65*t^37 + 9*q^64*t^38 + 2*q^65*t^36 - 18*q^64*t^37 + 26*q^63*t^38 + 8*q^64*t^36 -
27*q^63*t^37 + 5*q^62*t^38 - 4*q^63*t^36 + 2*q^62*t^37 + 2*q^61*t^38 + q^63*t^35 - 16*q^62*t^36 + 19*q^61*t^37 + 8*q^62*t^35 - 24*q^61*t^36 + 5*q^60*t^37 - 4*q^61*t^35 +
7*q^60*t^36 + q^59*t^37 + q^61*t^34 - 27*q^60*t^35 + 19*q^59*t^36 + 10*q^60*t^34 - 32*q^59*t^35 + 5*q^58*t^36 + q^60*t^33 - 10*q^59*t^34 + 2*q^58*t^35 - q^57*t^36 +
6*q^59*t^33 - 32*q^58*t^34 + 10*q^57*t^35 + 15*q^58*t^33 - 37*q^57*t^34 + 4*q^56*t^35 + q^58*t^32 + 5*q^57*t^33 - 4*q^56*t^34 + 4*q^57*t^32 - 23*q^56*t^33 + 6*q^55*t^34 +
17*q^56*t^32 - 28*q^55*t^33 + 3*q^54*t^34 + q^56*t^31 - 2*q^55*t^32 + 2*q^54*t^33 + 9*q^55*t^31 - 31*q^54*t^32 + 4*q^53*t^33 + 20*q^54*t^31 - 30*q^53*t^32 + 2*q^54*t^30 +
7*q^53*t^31 - 3*q^52*t^32 + 8*q^53*t^30 - 23*q^52*t^31 + 2*q^51*t^32 + 31*q^52*t^30 - 25*q^51*t^31 - q^52*t^29 + 15*q^51*t^30 - 3*q^50*t^31 + 9*q^51*t^29 - 14*q^50*t^30 +
3*q^49*t^31 - q^51*t^28 + 27*q^50*t^29 - 13*q^49*t^30 + q^50*t^28 + 6*q^49*t^29 - 2*q^48*t^30 + 6*q^49*t^28 - 13*q^48*t^29 - q^49*t^27 + 32*q^48*t^28 - 10*q^47*t^29 -
3*q^48*t^27 + 7*q^47*t^28 - 2*q^46*t^29 + 12*q^47*t^27 - 12*q^46*t^28 + q^45*t^29 - 5*q^47*t^26 + 36*q^46*t^27 - 8*q^45*t^28 - 6*q^46*t^26 + 9*q^45*t^27 + q^44*t^28 +
6*q^45*t^26 - 4*q^44*t^27 - 5*q^45*t^25 + 21*q^44*t^26 - 3*q^43*t^27 - 9*q^44*t^25 - 3*q^43*t^26 + 14*q^43*t^25 - 6*q^42*t^26 - 9*q^43*t^24 + 25*q^42*t^25 - 7*q^41*t^26 -
12*q^42*t^24 - 5*q^41*t^25 + q^40*t^26 - q^42*t^23 + 19*q^41*t^24 - 7*q^40*t^25 - 8*q^41*t^23 + 24*q^40*t^24 - 4*q^39*t^25 - 16*q^40*t^23 - 5*q^39*t^24 + 12*q^39*t^23 -
4*q^38*t^24 - 8*q^39*t^22 + 20*q^38*t^23 - 3*q^37*t^24 - 12*q^38*t^22 - 7*q^37*t^23 - 2*q^38*t^21 + 22*q^37*t^22 - 8*q^36*t^23 - 14*q^37*t^21 + 25*q^36*t^22 - q^35*t^23 -
14*q^36*t^21 - 5*q^35*t^22 + q^36*t^20 + 17*q^35*t^21 - 5*q^34*t^22 - 11*q^35*t^20 + 23*q^34*t^21 - q^33*t^22 - 19*q^34*t^20 - 2*q^33*t^21 + 3*q^34*t^19 + 19*q^33*t^20 -
5*q^32*t^21 - 13*q^33*t^19 + 25*q^32*t^20 - 18*q^32*t^19 - 2*q^31*t^20 + q^32*t^18 + 17*q^31*t^19 - 2*q^30*t^20 - 12*q^31*t^18 + 22*q^30*t^19 + q^31*t^17 - 24*q^30*t^18 -
q^29*t^19 + 4*q^30*t^17 + 14*q^29*t^18 - q^28*t^19 - 16*q^29*t^17 + 20*q^28*t^18 + 3*q^29*t^16 - 24*q^28*t^17 + q^27*t^18 + 3*q^28*t^16 + 8*q^27*t^17 - 12*q^27*t^16 +
15*q^26*t^17 + 2*q^27*t^15 - 23*q^26*t^16 + 3*q^26*t^15 + 7*q^25*t^16 - 19*q^25*t^15 + 11*q^24*t^16 + 8*q^25*t^14 - 25*q^24*t^15 + 6*q^24*t^14 - 9*q^23*t^14 + 7*q^22*t^15 +
6*q^23*t^13 - 20*q^22*t^14 + 6*q^22*t^13 + q^21*t^14 - 9*q^21*t^13 + 4*q^20*t^14 + 7*q^21*t^12 - 15*q^20*t^13 + 4*q^20*t^12 - q^19*t^13 + q^20*t^11 - 8*q^19*t^12 + 2*q^18*t^13
+ 9*q^19*t^11 - 14*q^18*t^12 + 10*q^18*t^11 - 4*q^17*t^11 + 7*q^17*t^10 - 8*q^16*t^11 + 8*q^16*t^10 - q^15*t^10 + 5*q^15*t^9 - 6*q^14*t^10 + 8*q^14*t^9 - q^14*t^8 - 2*q^13*t^9 +
7*q^13*t^8 - 2*q^12*t^9 + 7*q^12*t^8 - 4*q^12*t^7 - q^11*t^8 + 5*q^11*t^7 - q^10*t^8 + 6*q^10*t^7 - 5*q^10*t^6 - q^9*t^7 + 3*q^9*t^6 + 2*q^8*t^6 - 2*q^8*t^5 - q^7*t^6 + 4*q^7*t^5 -
2*q^7*t^4 + 2*q^6*t^5 - 3*q^6*t^4 + 2*q^5*t^4 - 2*q^5*t^3 - q^4*t^3 + q^3*t^3 - q^3*t^2 - q^2*t^2 - 1)/((q^26*t^15 - 1)*(q^14*t^8 + q^7*t^4 + 1)*(q^13*t^7 - 1)*(q^12*t^7 - 1)*(q^11*t^6
+ 1)*(q^11*t^6 - 1)^3*(q^9*t^5 - 1)*(q^8*t^5 - 1)*(q^7*t^4 - 1)^2*(q^4*t^3 - 1)*(q^3*t^2 - 1)*(q*t - 1)*(t - 1))
```

HOW DO YOU KNOW IT'S CORRECT?

Theorem (Voll - 2010)

$$\zeta_{L_p}(s)\big|_{p \to p^{-1}} = (-1)^n p^{\binom{n}{2} - ns} \zeta_{L_p}(s)$$

```
(ActualZ(t=1/t,q=1/q)/ActualZ).factor()

-q^10*t^5
```

- **Topological zeta function** (limit $p \to 1$) coincide with the one independently computed by Rossmann (2015).
- **Reduced zeta function** (another limit $p \to 1$) is consistent with expectations.
- A number of conjectures by Rossmann on poles at $t = 1$ (recently proved by Reunbrouck for classes of algebras containing Fil_4) hold for our formula.

Let us make a game!

State of a game: a list of `ToricDatum`

Possible moves:

- blow up between x_i and x_j
- change of variables
- some other tricks?

Let a Neural Network agent play it and give it reward based on the **weight** of TD's.

CURRENT PROGRESS

- New data structure: `FactoredToricDatum`:
  - better suited for our "games"
  - an action by $GL_n(\mathbb{Z})$ preserves the integral
  - specific blow-ups and changes of variables are integrated as methods
  - the structure itself improves performance (some types of singularities are automatically removed).
- some initial reinforcement learning experiments:
  - trained case-by-case
  - reduce the maximal weight
  - successful computations in some cases
  - new approaches are constantly "discovered"